A note on the distribution and density of the weighted sum of independent χ^2 distributed random variables

Peter Jez

Technical Report 2015-09 August 2015

Department of Computer Sciences

Jakob-Haringer-Straße 2
5020 Salzburg
Austria
www.cosy.sbg.ac.at

Technical Report Series

A note on the distribution and density of the weighted sum of independent χ^2 distributed random variables

Peter Jez; Department of Computer Sciences,University of Salzburg
email: peter.jez@aon.at

August 9, 2015

1 Introduction and main result

In this note we consider the probability distribution of a random variable

$$Y_N := \sum_{k=1}^{N} a_k^2 X_k$$

with $a_k \in \mathbb{R} \setminus \{0\}$ and $X_k \sim \chi^2_{n_k}$ independent random variables. No closed form is known for this distribution so far. The random variable Y_N is a generalization of the weighted sum of squares of i.i.d. $N(0,1)$ random variables. For this special case several computation methods were investigated. One method is the expansion into an infinite sum of independent χ^2 random variables, see [7], an other one is the inversion of the characteristic function, see [3]. For specific assumptions on the coefficients a_i and even degree n the distribution function can be expanded into a linear combination of χ^2 distributions [5]. Another approach is the approximation of with aid of a non central χ^2 distribution [4] and an approximation in terms of Laguerre expansions [1]. A series expansion of the distribution of the sum of weighted squares of independent $N(0,1)$ random variables has been provided by [6] where the coefficients are computed recursively. Our result is

Theorem 1. *Let $z > 0$,*

$$\varphi_N(a_1, \dots, a_N; z) := P(Y_N \le z)$$

and

$$F^{p_0;p_1,\dots,p_N}_{q_0;q_1,\dots,q_N}\left(\begin{array}{c} a_1,\dots a_{p_0}; b_1^{(1)},\dots,b_{p_1}^{(1)},\dots,b_1^{(N)},\dots,b_{p_N}^{(N)} \\ c_1,\dots c_{q_0}; d_1^{(1)},\dots,d_{p_1}^{(1)},\dots,d_1^{(N)},\dots,d_{p_N}^{(N)} \end{array}; x_1,\dots,x_N\right)$$

denote Srivastava's generalized hypergeometric function [8]. Let $J_n(x)$ be the Bessel function of first kind of degree n. Then we have the Neumann expansion

$$\varphi_N(a_1,\ldots,a_N;z)=\frac{1}{\prod_{p=1}^N|a_p|^{n_p}}\frac{1}{\Gamma\left(\frac{1}{2}\sum_{p=1}^N n_p+1\right)}\sum_{m=0}^\infty c_m J_{n+\sum_{p=1}^N n_p}\left(\sqrt{2z}\right)$$

with the coefficients

$$c_m=\frac{\left(2m+\sum_{p=1}^N n_p\right)\Gamma\left(m+\sum_{p=1}^N n_p\right)}{m!}F^{2;1,\ldots,1}_{1;0,\ldots,0}\left(\begin{array}{c}-m,m+\sum_{p=1}^N n_p;\frac{1}{2},\ldots,\frac{1}{2}\\ \frac{1}{2}\sum_{p=1}^N n_p+1;-\end{array};\frac{1}{a_1^2},\ldots,\frac{1}{a_n^2}\right).$$

By setting $n_k\equiv 1$ for $k=1,\ldots,N$ we obtain a Neumann expansion of the distribution of a weighted sum of the squares of independent $N(0,1)$ random variables:

Corollary 1. *Let $z>0$, $a_1,\ldots,a_N\in\mathbb{R}\setminus\{0\}$ and*

$$W_N:=\sum_{k=1}^N a_k^2X_k^2$$

with $X_k\sim N(0,1)$ i.i.d. random variables. Then we have the Neumann expansion

$$P(W_N\le z)=\frac{1}{\prod_{p=1}^N|a_p|}\frac{1}{\Gamma\left(\frac{N}{2}+1\right)}\sum_{m=0}^\infty c_m J_{n+N}\left(\sqrt{2z}\right)$$

with the coefficients

$$c_m=\frac{(2m+N)\Gamma(m+N)}{m!}F^{2;1,\ldots,1}_{1;0,\ldots,0}\left(\begin{array}{c}-m,m+N;\frac{1}{2},\ldots,\frac{1}{2}\\ \frac{N}{2}+1;-\end{array};\frac{1}{a_1^2},\ldots,\frac{1}{a_n^2}\right).$$

2 Proof of Theorem 1

For the proof of Theorem 1 we compute the distribution straight forward. Due to the well known fact that

$$P\left(\chi_m^2\le z\right)=\frac{1}{\Gamma\left(\frac{m}{2}\right)2^{\frac{m}{2}}}\int_0^z x^{\frac{m}{2}-1}e^{-\frac{x}{2}}dx$$

for $z>0$ we get

$$\varphi_N(a_1,\ldots,a_N;z)=\left(\prod_{j=1}^N\frac{1}{\Gamma\left(\frac{n_j}{2}\right)2^{\frac{n_j}{2}}}\right)\int_{\sum_{k=1}^N a_k^2x_k\le z}\left(\prod_{j=1}^N x_j^{\frac{n_j}{2}-1}e^{-\frac{x_j}{2}}\right)dx=$$

$$=\left(\prod_{j=1}^N\frac{z^{\frac{n_j}{2}}}{\Gamma\left(\frac{n_j}{2}\right)2^{\frac{n_j}{2}}|a_j|^{n_j}}\right)\int_{\{(y_1,\ldots,y_N):y_j\ge0,j=1,\ldots,N;y_1+\ldots+y_N\le1\}}\left(\prod_{j=1}^N y_j^{\frac{n_j}{2}-1}e^{-\frac{zy_j}{2a_j^2}}\right)dy.$$

We expand

$$\prod_{j=1}^{N} y_j^{\frac{n_j}{2}-1} e^{-\frac{zy_j}{2a_j^2}}$$

in a Taylor series and we get

$$\prod_{j=1}^{N} y_j^{\frac{n_j}{2}-1} e^{-\frac{zy_j}{2a_j^2}} = \sum_{k_1,\ldots,k_N=0}^{\infty} \frac{(-1)^{k_1+\ldots+k_n} z^{k_1+\ldots+k_N}}{k_1!\ldots k_N! \left(2a_1^2\right)^{k_1} \ldots \left(2a_N^2\right)^{k_N}} y_1^{k_1+\frac{n_1}{2}-1} \ldots y_N^{k_N+\frac{n_N}{2}-1}.$$

Due to the formula (see [2])

$$\int_{\{(y_1,\ldots,y_N):y_j\geq 0, j=1,\ldots,N; y_1+\ldots+y_N\leq 1\}} y_1^{q_1-1} \ldots y_N^{q_N-1} dy = \frac{\Gamma(q_1)\ldots\Gamma(q_N)}{\Gamma(1+q_1+\ldots+q_N)}$$

for $q_j > 0$ for $j = 1, \ldots, N$ we have the expansion

$$\varphi_N(a_1, \ldots, a_N; z) = \left(\prod_{j=1}^{N} \frac{z^{\frac{n_j}{2}}}{\Gamma\left(\frac{n_j}{2}\right) 2^{\frac{n_j}{2}} |a_j|^{n_j}}\right) \times$$

$$\times \sum_{k_1,\ldots,k_N=0}^{\infty} \frac{(-1)^{k_1+\ldots+k_n} z^{k_1+\ldots+k_N}}{k_1!\ldots k_N! \left(2a_1^2\right)^{k_1} \ldots \left(2a_N^2\right)^{k_N}} \frac{\Gamma\left(k_1+\frac{n_1}{2}\right)\ldots\Gamma\left(k_N+\frac{n_N}{2}\right)}{\Gamma\left(1+k_1+\ldots+k_N+\frac{1}{2}\sum_{p=1}^{N} n_p\right)} =$$

$$= \left(\prod_{j=1}^{N} \frac{z^{\frac{n_j}{2}}}{\Gamma\left(\frac{n_j}{2}\right) 2^{\frac{n_j}{2}} |a_j|^{n_j}}\right) \times$$

$$\times \frac{\Gamma\left(\frac{n_1}{2}\right)\ldots\Gamma\left(\frac{n_N}{2}\right)}{\Gamma\left(\frac{1}{2}\sum_{p=1}^{N} n_p+1\right)} \sum_{k_1,\ldots,k_N=0}^{\infty} \frac{\left(\frac{n_1}{2}\right)_{k_1}\ldots\left(\frac{n_N}{2}\right)_{k_N}}{\left(\frac{1}{2}\sum_{p=1}^{N} n_p+1\right)_{k_1+\ldots+k_N} k_1!\ldots k_N!} \left(-\frac{z}{2a_1^2}\right)^{k_1} \ldots \left(-\frac{z}{2a_N^2}\right)^{k_N}.$$

Remember the definition of the generalization of the hypergeometric function to n variables by Srivastava [8]: Let $\overrightarrow{a} = (a_1, \ldots, a_p)$, $\overrightarrow{b} = (b_1, \ldots, b_q)$ and $\overrightarrow{a_j} = \left(a_1^{(j)}, \ldots, a_{p_j}^{(j)}\right)$, $\overrightarrow{b_j} = \left(b_1^{(j)}, \ldots, b_{q_j}^{(j)}\right)$. Define the generalized Pochhammer symbol by

$$\begin{aligned}
\left(\overrightarrow{a}\right)_m &:= \prod_{k=1}^{p} (a_k)_m \\
\left(\overrightarrow{b}\right)_m &:= \prod_{k=1}^{q} (b_k)_m \\
\left(\overrightarrow{a_j}\right)_m &:= \prod_{k=1}^{p_j} \left(a_k^{(j)}\right)_m \\
\left(\overrightarrow{b_j}\right)_m &:= \prod_{k=1}^{q_j} \left(b_k^{(j)}\right)_m
\end{aligned}$$

Then Srivastava's generalized hypergeometric series is defined by

$$F^{p_0;p_1,\dots,p_n}_{q_0;q_1,\dots,q_n}\left(\begin{array}{c}\overrightarrow{a};\overrightarrow{a_1},\dots,\overrightarrow{a_n}\\ \overrightarrow{b};\overrightarrow{b_1},\dots,\overrightarrow{b_n}\end{array};x_1,\dots,x_n\right):=$$

$$:=\sum_{k_1,\dots,k_n=0}^{\infty}\frac{\left(\overrightarrow{a}\right)_{k_1+\dots+k_n}}{\left(\overrightarrow{b}\right)_{k_1+\dots+k_n}}\frac{x_1^{k_1}\dots x_n^{k_n}}{k_1!\dots k_n!}\prod_{j=1}^{n}\frac{\left(\overrightarrow{a_j}\right)_{k_j}}{\left(\overrightarrow{b_j}\right)_{k_j}}.$$

So we get

$$\varphi_N\left(a_1,\dots,a_N;z\right)=\frac{1}{\Gamma\left(\frac{1}{2}\sum_{j=1}^{N}n_j+1\right)}\left(\prod_{j=1}^{N}\frac{1}{|a_j|^{n_j}}\right)\times$$

$$\times\left(\frac{1}{2}z\right)^{\frac{1}{2}\sum_{j=1}^{N}n_j}F^{0;1,\dots,1}_{1;0,\dots,0}\left(\begin{array}{c}-;\frac{n_1}{2},\dots,\frac{n_N}{2}\\ \frac{1}{2}\sum_{j=1}^{N}n_j+1;-\end{array};-\frac{z}{2a_1^2},\dots-\frac{z}{2a_N^2}\right)=$$

$$=\frac{1}{\Gamma\left(\frac{1}{2}\sum_{j=1}^{N}n_j+1\right)}\left(\prod_{j=1}^{N}\frac{1}{|a_j|^{n_j}}\right)\left(\frac{1}{2}\sqrt{2z}\right)^{\sum_{j=1}^{N}n_j}F^{0;1,\dots,1}_{1;0,\dots,0}\left(\begin{array}{c}-;\frac{n_1}{2},\dots,\frac{n_N}{2}\\ \frac{1}{2}\sum_{j=1}^{N}n_j+1;-\end{array};-\frac{z}{2a_1^2},\dots-\frac{z}{2a_N^2}\right)$$

Srivastava [8] obtained the expansion

$$\left(\frac{1}{2}z\right)^{\lambda}F^{p_0;p_1,\dots,p_n}_{q_0;q_1,\dots,q_n}\left(\begin{array}{c}\overrightarrow{a_0};\overrightarrow{a_1},\dots,\overrightarrow{a_n}\\ \overrightarrow{b_0};\overrightarrow{b_1},\dots,\overrightarrow{b_n}\end{array};(-1)^l\left(\frac{w_1z}{2l}\right)^{2l},\dots,(-1)^l\left(\frac{w_nz}{2l}\right)^{2l}\right)=$$

$$=\sum_{m=0}^{\infty}\frac{(\lambda+2m)\,\Gamma(\lambda+m)}{m!}J_{\lambda+2m}(z)\,F^{2l+p_0;p_1,\dots,p_n}_{q_0;q_1,\dots,q_n}\left(\begin{array}{c}\Delta(l;-m)\ \Delta(l;\lambda+m)\,\overrightarrow{a_0};\overrightarrow{a_1},\dots,\overrightarrow{a_n}\\ \overrightarrow{b_0};\overrightarrow{b_1},\dots,\overrightarrow{b_n}\end{array};w_1^{2l},\dots,w_n^{2l}\right)$$

with the notation

$$\Delta(l;t)=\left(\frac{t}{l},\frac{t+1}{l},\dots,\frac{t+l-1}{l}\right).$$

Especially for $l=1$ we have $\Delta(1;t)=t$. We set $l=1$, $\lambda=\sum_{j=1}^{N}n_j$, $w_i^2=\frac{1}{a_i^2}$,$z=\sqrt{2x}$ and we get

$$\varphi_N\left(a_1,\dots,a_N;z\right)=\frac{1}{\Gamma\left(\frac{1}{2}\sum_{j=1}^{N}n_j+1\right)}\left(\prod_{j=1}^{N}\frac{1}{|a_j|^{n_j}}\right)\sum_{m=0}^{\infty}c_mJ_{\sum_{j=1}^{N}n_j+2m}\left(\sqrt{2z}\right)$$

with the coefficients

$$c_m=\frac{\left(2m+\sum_{j=1}^{N}n_j\right)\Gamma\left(m+\sum_{j=1}^{N}n_j\right)}{m!}F^{2;1,\dots,1}_{1;0,\dots,0}\left(\begin{array}{c}-m,\sum_{j=1}^{N}n_j+m;\frac{1}{2},\dots,\frac{1}{2}\\ \frac{1}{2}\sum_{j=1}^{N}n_j+1;-\end{array};\frac{1}{a_1^2},\dots,\frac{1}{a_n^2}\right).$$

So the proof of Theorem 1 is complete.

Remark 1. Due to the parameter $-m$ in the expansion coefficients c_m these coefficients are polynomials in $\left(\frac{1}{a_1^2},\dots,\frac{1}{a_N^2}\right)$ of degree m.

References

[1] A. Castano-Martinez and F. Lopez-Blazquez. Distribution of a sum of weighted noncentral chi-square variables. *TEST*, 14(2):397–415, 2005.

[2] R. F. Boisvert F. W. Olver, D. M. Lozier and C. W. Clark. *NIST Handbook of Mathematical Functions*. Cambridge Univerity Press, New York, 2010.

[3] A.H. Feiveson and F.C. Delaney. The distribution and properties of a weighted sum of chi squares. *National Aeronautics and Space Administration*, 1968.

[4] Y. Tang H. Liu and H. H. Zhang. A new chi-square approximation to the distribution of non-negativedefinite quadratic forms in non-central normal variables. *Computational Statistics and Data Analysis*, 53:853–856, 2009.

[5] H. Hotelling. Some new methods for distributions of quadratic forms. *Ann. Math. Stat.*, 19, 1948.

[6] H. Robbins. The distribution of a definite quadratic form. *Ann. Math. Statist.*, 19(2):266–270, 06 1948.

[7] H. Ruben. Probability content of regions under spherical normal distributions, I. *Ann. Math. Statist.*, 31(3):598–618, 09 1960.

[8] H M Srivastava. Neumann expansions for a certain class of generalised multiple hypergeometric series arising in physical and quantum chemical applications. *Journal of Physics A: Mathematical and General*, 20(4):847, 1987.