About a Weyl criterion for continous L^p functions

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Abstract

Numerical integration of functions $f : \mathbb{C}^s \to \mathbb{C}$ must be done with very high effort due to the curse of dimensionality. Therefor statistical methods like (Q)MC-methods are preferred. These methods are not so accurate as other but do not suffer so much under the increase of dimensionality. In this article we define a kind of uniform distribution of point sequences $\{z_k\}_{k\geq 1} \in \mathbb{C}^s$ and a criterion when they can be used for this integration method for special classes of functions (continous $L^p(\mathbb{C}^s)$ functions). A RKHS spanned by special Hermite functions is considered to get a Diaphony for these point sequences. This Diaphony gives another criterion for the uniform distribution.

1 Introduction

In (Q)MC-integration the distribution of the integration nodes $\{z_k\}_{k\geq 1}$ plays a crucial role for the accuracy of the results. In the following we consider MC integration of continous functions $f : \mathbb{C}^s \to \mathbb{C}$ with

$$\int_{\mathbb{C}^s} |f|^p dz < \infty$$

for $1 \leq p < \infty$. Let $\{z_k\}_{k\geq 1} \in \mathbb{C}^s$ and $\sigma \in L^q$ with

$$\frac{1}{p} + \frac{1}{q} = 1.$$

be a positive, normed density function. We investigate the functional

$$E_N(f) := \left| \frac{1}{N} \sum_{k=1}^{N} f(z_k) - \int_{\mathbb{C}^s} f(z)\sigma(z)dz \right| \tag{1}$$

A Weyl criterion is a condition on the point sequence to have $E_N(f) \to 0$ for $N \to \infty$ for a class of functions f. The special Hermite functions $\Phi_{\mu\nu}$ play an important role in the following therefor we recall the basic facts about this function sequence (see [7]):

Definition 1. *Let $x \in \mathbb{R}^s$ and $\Phi_\mu(x)$ with $\mu \in \mathbb{N}_0^s$ denote the real Hermite functions*

$$\Phi_\mu(x) = \Phi_\mu(x_1, \dots, x_s) = \prod_{j=1}^{s} \frac{H_{m_j}(x_j) \exp\left(-\frac{x_j^2}{2}\right)}{\sqrt{2_j^m m_j! \sqrt{\pi}}}$$

where H_m denotes the Hermite polynomial of degree m. For $z = (z_1, \dots, z_s) = (x_1 + iy_1, \dots, x_s + iy_s)$ the special Hermite function $\Phi_{\mu\nu}(z)$ is given by

$$\Phi_{\mu\nu}(z) = \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} \exp(ix.\xi)\, \Phi_\mu\left(\xi + \frac{y}{2}\right) \Phi_\nu\left(\xi - \frac{y}{2}\right) d\xi \qquad (2)$$

Remark 1. *The right hand side of (2) is the Fourier-Wigner transform $V(\Phi_m, \Phi_n)(z)$ (see [7]). So another representation of the special Hermite function $\Phi_{\mu\nu}$ is given by*

$$\Phi_{mn}(z) = V(\Phi_m, \Phi_n)(z).$$

From the definition it follows directly that the special Hermite functions are separable and they belong to Schwartz class. The special Hermite functions form a complete orthonormal system on $L^2(\mathbb{C}^s)$ and are eigenfunctions of the Hermite operator $-\Delta_z + \frac{1}{2}|z|^2$ with eigenvalues $|\mu| + |\nu| + s$. The special Hermite functions are related to the associated Laguerre functions: Let L_k^{s-1} denote the associated Laguerre polynomial and let

$$\varphi_k(z) := L_k^{s-1}\left(\frac{1}{2}|z|^2\right) \exp\left(-\frac{1}{4}|z|^2\right)$$

Then we have the identity [7]:

$$\sum_{|\nu|=k} \Phi_{\nu\nu}(z) = \varphi_k(z)$$

Given any function $f(z) \in L^p(\mathbb{C}^s)$ the formal expansion in special Hermite functions

$$f(z) = \sum_{\mu,\nu \in \mathbb{N}_0^s} \left(\int_{\mathbb{C}^s} f(z)\overline{\Phi_{\mu\nu}} dz\right) \Phi_{\mu\nu}(z)$$

can be written in the form

$$f(z) = \frac{1}{(2\pi)^s} \sum_{k=0}^{\infty} f \times \varphi_k(z)$$

where $\times$ denotes the twisted convolution:

$$f \times g(z) := \int_{\mathbb{C}^s} f(z-w) g(w) \exp\left(\frac{i}{2}\Im(z\overline{w})\right) dw$$

In general the expansion does not converge to f unless $f \in L^2$. The Riesz means given by

$$S_R^\delta f(z) = \frac{1}{(2\pi)^s} \sum_{k \geq 0; 2k+s \leq R} \left(1 - \frac{2k+n}{R}\right)^\delta f \times \varphi_k(z)$$

converge to $f \in L^p$ with additional assumptions on δ (see [7]). The special Hermite functions are closely related to the complex Hermite polynomials $H_{m,n}(z,\overline{z})$ introduced by [6] (see [2]):

$$\Phi_{mn}(z) = \frac{(-1)^n}{\sqrt{2m!n!\pi}} \exp\left(-\frac{|z|^2}{4}\right) H_{m,n}\left(\frac{z}{\sqrt{2}}, \frac{\overline{z}}{\sqrt{2}}\right) \tag{3}$$

Remark 2. *From this identity and the generating function of the complex Hermite polynomials (see [5])*

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{n!m!} H_{m,n}(z,\overline{z}) = \exp(-uv + zu + \overline{z}v)$$

we obtain the following generating function of the special Hermite functions:

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{\sqrt{m!n!}} \Phi_{mn}(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{8}|z|^2 + uv + \frac{1}{\sqrt{2}}zu - \frac{1}{\sqrt{2}}\overline{z}v\right)$$

The complex Hermite plynomials can be generalized to s dimensions in the same way as the real Hermite polynomials. The paper is organized as follows: In section 2 a criterion for a uniform distribution of a point sequence in s-dimensional complex space based on special Hermite functions is formulated and proven based on Riesz means. In section 3 we introduce a RKHS spanned by special Hermite functions which allow the compuation of a Diaphony and gives a criterion for uniform distribution with respect to a circulary symmetric complex Gaussian distribution.

2 A Weyl criterion for continous L^p functions

We start with the following

Definition 2. *A sequence of points $\{z_k\}_{k\geq 1} \in \mathbb{C}^s$ is called uniformly distributed with respect to the weight function $\sigma \in L^q(\mathbb{C}^s)$ if we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(z_k) = \int_{\mathbb{C}^s} f(z)\sigma(z)dz \tag{4}$$

for all continous functions $f(z)$ with

$$f(z) \in L^p(\mathbb{C}^s)$$

where $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Our aim is to prove the following:

Theorem 1. *Let $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. A sequence of points $\{z_k\}_{k\geq 1} \in \mathbb{C}^s$ is uniformly distributed with respect to the weight function $\sigma \in L^q(\mathbb{C}^s)$ if and only if we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \Phi_{\mu\nu}(z_k) = \int_{\mathbb{C}^s} \Phi_{\mu\nu}(z)\sigma(z)dz \tag{5}$$

for all $\mu, \nu \in \mathbb{N}_0^s$ where $\Phi_{\mu\nu}$ denotes the special Hermite functions.

We start with the following

Lemma 1. *A sequence of points* $\{z_k\}_{k\geq 1} \in \mathbb{C}^s$ *is uniformly distributed with respect to the weight function* $\sigma \in L^q(\mathbb{C}^s)$ *if and only if we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(z_k) = \int_{\mathbb{C}^s} f(z)\sigma(z)dz \tag{6}$$

for all $f \in C_0^\infty(\mathbb{C}^s)$.

Proof. If the point sequence $\{z_k\}_{k\geq 1} \in \mathbb{C}^s$ is uniformly distributed with respect to σ the we have the property (4) for all $f \in C_0^\infty(\mathbb{C}^s)$ because $C_0^\infty(\mathbb{C}^s) \subset L^p(\mathbb{C}^s) \cap C(\mathbb{C}^s)$. Let $\varepsilon > 0$ and $f \in L^p$, f continous, be given. Then there is an $R_1(\varepsilon)$ with

$$\int_{\mathbb{C}^s\setminus\{z:|z|\leq R\}} |f|^p < \varepsilon \tag{7}$$

for all $R > R_1(\varepsilon)$ and (due to the continuity) an $R_2(\varepsilon)$ with $|f| < \varepsilon$ for all $|z| > R$ for all $R > R_2(\varepsilon)$. Let $R(\varepsilon) = \max(R_1(\varepsilon), R_2(\varepsilon))$. Let $f_\varepsilon(z) = f(z)$ for $|z| \leq R(\varepsilon)$ and 0 elsewhere. We consider the mollification $h_\delta(z) \in C_0^\infty(\mathbb{C}^s)$ of f_ε. It is well known that the mollification of a continous L^p function g converges uniformly to g on compact sets for $\delta \to 0$ (see [4]). This means that there is a $\delta_0(\varepsilon)$ with

$$\max_{|z|\leq R(\varepsilon)} |f_\varepsilon - h_\delta| < \varepsilon \tag{8}$$

for all δ with $|\delta| < \delta_0(\varepsilon)$. Due to the continuity of f it is uniformly continous on compact sets and therefor we have a $\delta_1(\varepsilon)$ with the property that

$$|f_\varepsilon| < 2\varepsilon \tag{9}$$

for $R(\varepsilon) - \delta < |z|$ for all $\delta < \delta_1(\varepsilon)$. Let $\delta(\varepsilon) = \frac{1}{2}\min(\delta_0(\varepsilon), \delta_1(\varepsilon))$. Then we have $|h_\delta| < 2\varepsilon$ for $R(\varepsilon) - \delta < |z|$ for all $\delta < \delta(\varepsilon)$ and we get the estimation:

$$E_N(f) = \left| \frac{1}{N}\sum_{k=1}^{N} f(z_k) - \int_{\mathbb{C}^s} f(z)\sigma(z)dz \right| \leq$$

$$\leq \left| \frac{1}{N} \sum_{k=1;|z_k|>R(\varepsilon)}^{N} f(z_k) \right| + \tag{10}$$

$$+ \left| \frac{1}{N} \sum_{k=1;|z_k|\leq R(\varepsilon)}^{N} f(z_k) - \frac{1}{N} \sum_{k=1;|z_k|\leq R(\varepsilon)}^{N} h_\delta(z_k) \right| + \tag{11}$$

$$+ \left| \frac{1}{N} \sum_{k=1;|z_k|>R(\varepsilon)}^{N} h_\delta(z_k) \right| + \tag{12}$$

$$+ \left| \frac{1}{N}\sum_{k=1}^{N} h_\delta(z_k) - \int_{\mathbb{C}^s} h_\delta(z)\sigma(z)dz \right| + \tag{13}$$

$$+\left|\int_{\mathbb{C}^s} h_\delta(z)\sigma(z)dz - \int_{\mathbb{C}^s} f(z)\sigma(z)dz\right| \qquad (14)$$

The term (10) is less than ε because of the choice of $R(\varepsilon)$. Term (11) can be estimated in the following way:

$$\left|\frac{1}{N}\sum_{k=1;|z_k|\le R(\varepsilon)}^{N} f(z_k) - \frac{1}{N}\sum_{k=1;|z_k|\le R(\varepsilon)}^{N} h_\delta(z_k)\right| \le$$

$$\le \left|\frac{1}{N}\sum_{k=1;|z_k|\le R(\varepsilon)}^{N} f(z_k) - \frac{1}{N}\sum_{k=1;|z_k|\le R(\varepsilon)}^{N} f_\varepsilon(z_k)\right| +$$

$$+\left|\frac{1}{N}\sum_{k=1;|z_k|\le R(\varepsilon)}^{N} f_\varepsilon(z_k) - \frac{1}{N}\sum_{k=1;|z_k|\le R(\varepsilon)}^{N} h_\delta(z_k)\right|$$

The first term is equal to 0 because of the construction of f_ε. The second term is less than ε because of (8). Term (12) is less than 2ε due to the choice of δ. Term (14) can be estimated by Hölders inequality:

$$\left|\int_{\mathbb{C}^s} h_\delta(z)\sigma(z)dz - \int_{\mathbb{C}^s} f(z)\sigma(z)dz\right| \le$$

$$\le \left|\int_{\mathbb{C}^s} h_\delta(z)\sigma(z)dz - \int_{\mathbb{C}^s} f_\varepsilon(z)\sigma(z)dz\right| +$$

$$+\left|\int_{\mathbb{C}^s} f_\varepsilon(z)\sigma(z)dz - \int_{\mathbb{C}^s} f(z)\sigma(z)dz\right|$$

$$\le \|h_\delta - f_\varepsilon\|_p . \|\sigma\|_q + \varepsilon . \|\sigma\|_q .$$

The fact that $h_\delta \to f_\varepsilon$ in L^p (see [4]) and the previous results we have

$$E_N(f) \le 4\varepsilon + \|h_\delta - f_\varepsilon\|_p . \|\sigma\|_q + \varepsilon . \|\sigma\|_q +$$

$$+\left|\frac{1}{N}\sum_{k=1} h_\delta(z_k) - \int_{\mathbb{C}^s} h_\delta(z)\sigma(z)dz\right|$$

which gives the desired result for $\delta < \delta(\varepsilon)$. □

Now we continue with the proof of the theorem:

Proof. Let $\{z_k\}_{k\ge 1} \in \mathbb{C}^s$ be uniform distributed with respect to $\sigma \in L^q$. The special Hermite functions $\Phi_{\mu\nu}$ are continous and $\Phi_{\mu\nu} \in L^p$ for $p \ge 1$. Due to the uniform distribution of the z_k we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \Phi_{\mu\nu}(z_k) = \int_{\mathbb{C}^s} \Phi_{\mu\nu}(z)\sigma(z)dz$$

Now we assume (5). According to the lemma it is sufficient to prove the theorem for $f(z) \in C_0^\infty(\mathbb{C}^s)$. Let $\varepsilon > 0$ and an arbitrary $f(z) \in C_0^\infty(\mathbb{C}^s)$ be given. We consider the functional

$$E_N(f) = \left|\frac{1}{N}\sum_{k=1}^{N} f(z_k) - \int_{\mathbb{C}^s} f(z)\sigma(z)dz\right|$$

From [7] it is well known that the Riesz-Means $S_R^\delta f(z)$ converge to f in L^p-norm for $1 \le p < \infty$ for $\delta > s - \frac{1}{3}$. Now let

$$K = \{z \in \mathbb{C}^s \, ||z| \le r_K\} \subset \mathbb{C}^s$$

be given with $suppf \subset K$. Then we get

$$E_N(f) = \left| \frac{1}{N} \sum_{k=1}^{N} f(z_k) - \int_{\mathbb{C}^s} f(z)\sigma(z)dz \right| \le$$

$$\le \left| \frac{1}{N} \sum_{k=1;z_k \in K}^{N} f(z_k) - \frac{1}{N} \sum_{k=1;z_k \in K}^{N} (S_R^\delta f)(z_k) \right| + \tag{15}$$

$$+ \left| \frac{1}{N} \sum_{k=1;z_k \notin K}^{N} (S_R^\delta f)(z_k) \right| + \tag{16}$$

$$+ \left| \frac{1}{N} \sum_{k=1}^{N} (S_R^\delta f)(z_k) - \int_{\mathbb{C}^s} (S_R^\delta f)(z)\sigma(z)dz \right| + \tag{17}$$

$$+ \left| \int_{\mathbb{C}^s} (S_R^\delta f)(z)\sigma(z)dz - \int_{\mathbb{C}^s} f(z)\sigma(z)dz \right| \tag{18}$$

At first we consider term (15): There is an $R_1(\varepsilon)$ with the property that

$$\left\| S_R^\delta f(z) - f(z) \right\|_\infty < \varepsilon$$

for $z \in K$ and $R > R_1(\varepsilon)$. Term (18) can be estimated by Holders inequality:

$$\left| \int_{\mathbb{C}^s} (S_R^\delta f)(z)\sigma(z)dz - \int_{\mathbb{C}^s} f(z)\sigma(z)dz \right| \le$$

$$\le \left\| S_R^\delta f - f \right\|_p \|\sigma\|_q .$$

By the fact that the Riesz means converge to f for $\delta > s - \frac{1}{3}$ for $R \to \infty$ (see [7]) there is an $R_2(\varepsilon)$ to have $\left\| S_R^\delta f - f \right\|_p < \varepsilon$ for all $R > R_2(\varepsilon)$. For the term (16) we use the following bound for the Riesz kernel (see [7]):

$$\left| s_R^\delta(z) \right| \le C \frac{R^s}{\left(1 + R^{\frac{1}{2}} |z|\right)^{\delta + s + \frac{1}{3}}} \tag{19}$$

Now consider $(S_R^\delta f)(z)$ for $z \notin K$:

$$\left| (S_R^\delta f)(z) \right| \le C R^s \int_{\mathbb{C}^s} \frac{|f(w)|}{\left(1 + R^{\frac{1}{2}} |z - w|\right)^{\delta + s + \frac{1}{3}}} dw \le$$

$$\le C \frac{R^s}{R^{\frac{1}{2}\left(s + \delta + \frac{1}{3}\right)}} \int_{\mathbb{C}^s} \frac{|f(w)|}{(|z - w|)^{\delta + s + \frac{1}{3}}} dw \le$$

$$\le C \frac{R^s}{R^{\frac{1}{2}\left(s + \delta + \frac{1}{3}\right)}} \frac{1}{d^{\left(s + \delta + \frac{1}{3}\right)}} \|f\|_1$$

where $d = \min_{z \notin K, w \in supp f} |z - w| > 0$ according to our assumption on K. So for $\delta > s - \frac{1}{3}$ we have $S_R^\delta f(z) = O\left(\frac{1}{R^\eta}\right)$ for an $\eta > 0$ and therefor we can find an $R_3(\varepsilon)$ to have $|S_R^\delta f(z)| < \varepsilon$ for all $R > R_3(\varepsilon)$. Now let $R > R(\varepsilon) := \max(R_1(\varepsilon), R_2(\varepsilon), R_3(\varepsilon))$. From the following we have

$$E_N(f) \leq \varepsilon + \varepsilon \|\sigma\|_q + \left| \frac{1}{N} \sum_{k=1}^{N} (S_R^\delta) f(z_k) - \int_{\mathbb{C}^s} (S_R^\delta) f(z) \sigma(z) dz \right| + \varepsilon$$

$$= \varepsilon(2 + \|\sigma\|_q) + \left| \frac{1}{N} \sum_{k=1}^{N} (S_R^\delta) f(z_k) - \int_{\mathbb{C}^s} (S_R^\delta) f(z) \sigma(z) dz \right|.$$

Due to assumption (5) we can find an index $N(R(\varepsilon), \varepsilon)$ to have

$$\left| \frac{1}{N} \sum_{k=1}^{N} \Phi_{\mu\nu}(z_k) - \int_{\mathbb{C}^s} \Phi_{\mu\nu}(z) \sigma(z) dz \right| < \frac{\varepsilon}{R(\varepsilon)}$$

for all μ, ν with $|\mu|, |\nu| < R(\varepsilon)$ for $N > N(R(\varepsilon), \varepsilon)$. □

Identity (3) gives a criterion for uniform distribution with respect to σ based on complex Hermite polynomials:

Corollary 1. *Let $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. A sequence of points $\{z_k\}_{k \geq 1} \in \mathbb{C}^s$ is uniformly distributed with respect to the weight function $\sigma \in L^q(\mathbb{C}^s)$ if and only if we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} H_{\mu,\nu}(z_k) \exp\left(-\frac{|z|^2}{4}\right) = \int_{\mathbb{C}^s} H_{\mu,\nu}(z) \exp\left(-\frac{|z|^2}{4}\right) \sigma(z) dz \tag{20}$$

for all $\mu, \nu \in \mathbb{N}_0^s$ where $H_{\mu,nu}$ denotes the complex Hermite polynomial in s dimensions.

3 A reproducing kernel Hilbert space spanned by special Hermite functions

From the previous section we know that the distribution properties of a point sequence $\{z_k\}_{k \geq 1}$ can be checked with the special Hermite functions. In this section we compute the diaphony in a reproducing kernel Hilbert space spanned by special Hermite functions which gives another criterion for uniform distribution with respect to a circular symmetric Gaussian distribution. Let $\alpha = (\alpha_1, \ldots, \alpha_s)$, $|\alpha_j| < 1$, $\beta = (\beta_1, \ldots, \beta_s)$, $|\beta_j| < 1$. For $\nu = (n_1, \ldots, n_s) \in \mathbb{N}_0^s$ we write

$$\alpha^\nu = \alpha_1^{n_1} \ldots \alpha_s^{n_s}. \tag{21}$$

Let

$$H_{\alpha,\beta} := \left\{ f : \mathbb{C}^s \to \mathbb{C} | \qquad f(z) = \sum_{\mu,\nu \in \mathbb{N}_0^s} a_{\mu\nu} \alpha^\nu \beta^\mu \Phi_{\mu\nu}(z), \sum_{\mu,\nu \in \mathbb{N}_0^s} |a_{\mu\nu}|^2 < \infty \right\}. \tag{22}$$

We define a scalar product on $H_{\alpha,\beta}$ by

$$\left\langle \alpha^\mu \beta^\nu \Phi_{\mu\nu}, \alpha^\xi \beta^\eta \Phi_{\xi\eta} \right\rangle = \delta_{\mu\xi}\delta_{\nu\eta}. \tag{23}$$

Let

$$f(z) = \sum_{\mu,\nu\in\mathbb{N}_0^s} a_{\mu\nu}\Phi_{\mu\nu}(z)\alpha^\mu\beta^\nu$$

be an arbitrary element of $H_{\alpha,\beta}$. For the following we state the following generating function of the special Hermite functions (another generating function is derived in [2])

Lemma 2. *Let $0 < |\alpha|, |\beta| < 1$ and $z, w \in \mathbb{C}$. Then we have*

$$\sum_{n,k=0}^{\infty} \Phi_{nk}(z)\overline{\Phi_{nk}(w)}\alpha^{2n}\beta^{2k} = \tag{24}$$

$$= \frac{1}{2\pi\left(1-\alpha^2\beta^2\right)} \exp\left(-\left(|z|^2 + |w|^2\right) \frac{1+\alpha^2\beta^2}{4\left(1-\alpha^2\beta^2\right)} + \frac{\beta^2 z\overline{w} + \alpha^2 \overline{z} w}{2(1-\alpha^2\beta^2)} \right)$$

Proof. Reordering of (24) gives

$$\sum_{n,k=0}^{\infty} \Phi_{nk}(z)\overline{\Phi_{nk}(w)}\alpha^{2n}\beta^{2k} =$$

$$= \sum_{u=0}^{\infty} \alpha^{2u} \sum_{k=0}^{\infty} \Phi_{u+k,k}(z)\overline{\Phi_{u+k,k}(w)}\alpha^{2k}\beta^{2k} +$$

$$+ \sum_{u=0}^{\infty} \beta^{2u} \sum_{k=0}^{\infty} \Phi_{k,u+k}(z)\overline{\Phi_{k,u+k}(w)}\alpha^{2k}\beta^{2k}$$

By the well known formulas (see [7]) $\Phi_{kn}(z) = \overline{\Phi_{kn}(-z)}$ we have

$$\sum_{n,k=0}^{\infty} \Phi_{nk}(z)\overline{\Phi_{nk}(w)}\alpha^{2n}\beta^{2k} =$$

$$= \sum_{u=0}^{\infty} \alpha^{2u} \sum_{k=0}^{\infty} \Phi_{u+k,k}(z)\overline{\Phi_{u+k,k}(w)}\alpha^{2k}\beta^{2k} +$$

$$+ \sum_{u=0}^{\infty} \beta^{2u} \sum_{k=0}^{\infty} \overline{\Phi_{u+k,k}(-z)}\Phi_{u+k,k}(-w)\alpha^{2k}\beta^{2k}$$

From the formula (L_n^k denotes the associated Laguerre polynomial, see [7])

$$\Phi_{u+k,k}(z) = \frac{1}{\sqrt{2\pi}} \left(\frac{k!}{(u+k)!} \right)^{\frac{1}{2}} \left(\frac{i}{\sqrt{2}} \right)^u \overline{z}^u L_k^u \left(\frac{1}{2}|z|^2 \right) \exp\left(-\frac{1}{4}|z|^2 \right) \tag{25}$$

and the Poisson kernel formula for associated Laguerre functions

$$\sum_{k=0}^{\infty} \frac{k!}{(k+u)!} L_k^u(x) L_k^u(y) w^k = \frac{1}{(xyw)^{\frac{u}{2}}} \exp\left(-\frac{(x+y)w}{1-w} \right) I_u \left(\frac{2(xyw)^{\frac{1}{2}}}{1-w} \right) \tag{26}$$

and the relation $I_n(z) = I_{-n}(z)$ we get the formula

$$\sum_{n,k=0}^{\infty} \Phi_{nk}(z)\overline{\Phi_{nk}(w)}\alpha^{2n}\beta^{2k} = \tag{27}$$

$$= \frac{1}{2\pi(1-\alpha^2\beta^2)} \exp\left(-\left(|z|^2+|w|^2\right) \frac{1+\alpha^2\beta^2}{4\,(1-\alpha^2\beta^2)} \right) \times$$
$$\sum_{u=-\infty}^{\infty} I_u\left(\frac{|zw\alpha\beta|}{1-\alpha^2\beta^2}\right)\left(\frac{z\overline{w}|\beta|}{|zw\alpha|}\right)^u$$

From the generating function of the Bessel functions of imaginary argument (see [3])

$$\exp\left(\frac{1}{2}z\left(t+\frac{1}{t}\right)\right) = \sum_{u=-\infty}^{\infty} I_u(z)t^u$$

we get the formula

$$\sum_{n,k=0}^{\infty} \Phi_{nk}(z)\overline{\Phi_{nk}(w)}\alpha^{2n}\beta^{2k} =$$

$$= \frac{1}{2\pi\,(1-\alpha^2\beta^2)} \exp\left(-\left(|z|^2+|w|^2\right) \frac{1+\alpha^2\beta^2}{4\,(1-\alpha^2\beta^2)} \right) \exp\left(\frac{\beta^2 z\overline{w}+\alpha^2\overline{z}w}{2(1-\alpha^2\beta^2)}\right).$$

□

Remark 3. *We observe the following upper bound for*

$$\left|\sum_{n,k=0}^{\infty} \Phi_{nk}(z)\overline{\Phi_{nk}(w)}\alpha^{2n}\beta^{2k}\right|$$

by Cauchy Schwarz inequality:

$$\left|\sum_{n,k=0}^{\infty} \Phi_{nk}(z)\overline{\Phi_{nk}(w)}\alpha^{2n}\beta^{2k}\right| \le$$

$$\le \left(\sum_{n,k=0}^{\infty} |\Phi_{nk}(z)|^2\,\alpha^{2n}\right)^{\frac{1}{2}} \left(\sum_{n,k=0}^{\infty} |\Phi_{nk}(w)|^2\,\beta^{2k}\right)^{\frac{1}{2}} =$$
$$= \frac{1}{2\pi\,(1-\alpha^2\beta^2)} \exp\left(-\left(|z|^2+|w|^2\right) \frac{1+\alpha^2\beta^2-\alpha^2-\beta^2}{2\,(1-\alpha^2\beta^2)} \right) =$$
$$= \frac{1}{2\pi\,(1-\alpha^2\beta^2)} \exp\left(-\left(|z|^2+|w|^2\right) \frac{(1-\alpha^2)(1-\beta^2)}{4\,(1-\alpha^2\beta^2)} \right) =$$
$$= C\exp\left(-D\left(|z|^2+|w|^2\right)\right) \tag{28}$$

with positive constants C, D due to the fact that $1+xy-x-y>0$ for $0 \le x, y < 1$.

Identity (3) and Lemma 2 give a Poisson formula for the complex Hermite polynomials:

Corollary 2. *Let* $|\alpha|, |\beta| < 1$*. Then we have*

$$\sum_{m,n=0}^{\infty} \frac{H_{m,n}(z,\overline{z})\overline{H_{m,n}(w,\overline{w})}}{m!n!}\alpha^{2m}\beta^{2n} =$$

$$= \frac{1}{1-\alpha^2\beta^2}\exp\left(-\frac{\alpha^2\beta^2}{1-\alpha^2\beta^2}\left(|z|^2+|w|^2\right)+\frac{\beta^2 z\overline{w}+\alpha^2\overline{z}w}{1-\alpha^2\beta^2}\right).$$

Due to the separability of the special Hermite functions we have also the closed form for $s > 1$:

$$K(z,w) = \tag{29}$$

$$= \prod_{j=1}^{s}\left[\frac{1}{2\pi\left(1-\alpha_j^2\beta_j^2\right)}\exp\left(-\left(|z_j|^2+|w_j|^2\right)\frac{1+\alpha_j^2\beta_j^2}{4\left(1-\alpha_j^2\beta_j^2\right)}\right)\times\right.$$

$$\left.\exp\left(\frac{\beta_j^2 z_j\overline{w_j}+\alpha_j^2\overline{z_j}w_j}{2(1-\alpha_j^2\beta_j^2}\right)\right].$$

We consider an arbitrary function $f \in H_{\alpha\beta}$:

$$f(z) = \sum_{\mu,\nu\in\mathbb{N}_0^s} a_{\mu\nu}\Phi_{\mu\nu}(z)\alpha^{\mu}\beta^{\nu}. \tag{30}$$

The norm of f induced by the scalar product (23) is given by

$$\|f\| = \left(\sum_{\mu,\nu\in\mathbb{N}_0^s}|a_{\mu\nu}|^2\right)^{\frac{1}{2}}.$$

We have also the following estimation:

$$|f(z)| = \left|\sum_{\mu,\nu\in\mathbb{N}_0^s} a_{\mu\nu}\Phi_{\mu\nu}(z)\alpha^{\mu}\beta^{\nu}\right| \leq \|f\|.\left(\sum_{\mu,\nu\in\mathbb{N}_0^s}|\Phi_{\mu\nu}(z)|^2\alpha^{2\mu}\beta^{2\nu}\right)^{\frac{1}{2}}.$$

This and Remark 3 means that all evaluation functionals are continous which is the criterion that $H_{\alpha,\beta}$ is a reproducing kernel Hilbert space (see [1]). The kernel is given by

$$K(z,w) = \sum_{\mu,\nu\in\mathbb{N}_0^s}\Phi_{\mu\nu}(z)\overline{\Phi_{\mu\nu}(w)}\alpha^{2\mu}\beta^{2\nu}. \tag{31}$$

and we have the representation

$$f(z) = \left\langle f, \sum_{\mu,\nu\in\mathbb{N}_0^s}\Phi_{\mu\nu}(.)\overline{\Phi_{\mu\nu}(z)}\alpha^{\mu}\beta^{\nu}\right\rangle.$$

In the following we investigate the (Q)MC integration error with aid of the RKHS property of $H_{\alpha.\beta}$ (analog to [8]).

3.1 MC Integration on $H_{\alpha,\beta}$

Let $f \in H_{\alpha,\beta}$ and let $\{z_k\}_{k\geq 1} \in \mathbb{C}^s$ be a sequence of points and let

$$w(z) : \mathbb{C}^s \to \mathbb{R}, w(z) = w(z_1, \ldots, z_s) = \tag{32}$$

$$= \frac{1}{(4\pi)^s} \exp\left(-\frac{1}{4}\left(|z_1|^2 + \ldots + |z_s|^2\right)\right)$$

We investigate the integration error

$$E_N(f) := \left| \frac{1}{N} \sum_{k=1}^{N} f(z_k) - \int_{\mathbb{C}^s} f(z)w(z)dz \right| \tag{33}$$

Remark 4. *The weight function is related to the special Hermite function* $\Phi_{00}(z)$ *by*

$$w(z) = \left(\frac{1}{2\sqrt{2\pi}}\right)^s \Phi_{00}(z).$$

By the reproducing kernel property and Cauchy Schwarz inequality we get a Hlawka Koksma estimate for $E_N(f)$:

$$E_N(f) := \left| \frac{1}{N} \sum_{k=1}^{N} f(z_k) - \int_{\mathbb{C}^s} f(z)w(z)dz \right| \leq$$

$$\leq \|f\|. \left(\frac{1}{N^2} \sum_{k,l=1}^{N} K(z_k, z_l) - \frac{2}{N(2\sqrt{2\pi})^{\frac{s}{2}}} \sum_{k=1}^{N} \Phi_{00}(z_k) + \frac{1}{(8\pi)^s} \right)^{\frac{1}{2}} =$$

$$= \|f\|. \left(\frac{1}{N^2} \sum_{k,l=1}^{N} K(z_k, z_l) - \frac{2}{N(4\pi)^s} \sum_{k=1}^{N} \exp\left(-\frac{1}{4}|z_k|^2\right) + \frac{1}{(8\pi)^s} \right)^{\frac{1}{2}} \tag{34}$$

The term independent from f

$$D_N(z_k) := \tag{35}$$

$$\left(\frac{1}{N^2} \sum_{k,l=1}^{N} K(z_k, z_l) - \frac{2}{N(4\pi)^s} \sum_{k=1}^{N} \exp\left(-\frac{1}{4}|z_k|^2\right) + \frac{1}{(8\pi)^s} \right)^{\frac{1}{2}}$$

is the Diaphony $D_N(z_k)$ of the point sequence $\{z_k\}_{k\geq 1}$ (see [8]). The Diaphony (35) gives a criterion for the uniform Distribution of the point sequence $\{z_k\}_{k\geq 1}$ with respect to $w(z)$:

Theorem 2. *Let* $\{z_k\}_{k\geq 1} \in \mathbb{C}^s$ *be a sequence of points. The points are uniformly distributed with respect to the weight function* $w(z)$ *defined by (32) if and only if*

$$\lim_{N\to\infty} D_N(z_k) = 0 \tag{36}$$

Proof. Assume $D_N(z_k) \to 0$ for $N \to \infty$. Due to (34) and

$$\|\Phi_{\mu\nu}\| = \frac{1}{\alpha^\mu \beta^\nu}$$

we have $E_N(\Phi_{\mu\nu}) \to 0$. From Theorem 1 we have the uniform distribution of the sequence z_k. Now we assume that the sequence $\{z_k\}$ is uniformly distributed with respect to w and let $\varepsilon > 0$ be given. There is an $R(\varepsilon)$ with $C\exp(-DR^2) < \varepsilon$ for all $R > R(\varepsilon)$. C, D are the constants from Remark 3. Fix an $R > R(\varepsilon)$. Let

$$K_R := \{z \in \mathbb{C}^s \,||z| \le R\}$$

Let $I_M := \{0, 1, \ldots, M\}$. Due to the compactness of K_R and the uniform convergence of the series expansion of the kernel on compact subsets there is an index $M(\varepsilon)$ with

$$\left| \sum_{(\mu,\nu)\notin I_M^s \times I_M^s} \Phi_{\mu\nu}(z)\overline{\Phi_{\mu\nu}(w)}\alpha^{2\mu}\beta^{2\nu} \right| < \varepsilon$$

for all $M > M(\varepsilon)$ and all $z, w \in K_R$. We fix an $M > M(\varepsilon)$ and we get

$$D_N^2(z_k) = \frac{1}{N^2}\sum_{k,l=1}^{N}\sum_{(\mu,\nu)\in I_M^s \times I_M^s} \Phi_{\mu\nu}(z_k)\overline{\Phi_{\mu\nu}(z_l)}\alpha^{2\mu}\beta^{2\nu}$$

$$-\frac{2}{N(4\pi)^s}\sum_{k=1}^{N}\exp\left(-\frac{1}{4}|z_k|^2\right) + \frac{1}{(8\pi)^s} +$$

$$+\frac{1}{N^2}\sum_{k,l=1}^{N}\sum_{(\mu,\nu)\notin I_M^s \times I_M^s} \Phi_{\mu\nu}(z_k)\overline{\Phi_{\mu\nu}(z_l)}\alpha^{2\mu}\beta^{2\nu} =$$

$$= \left| \frac{1}{N}\sum_{k=1}^{N}\Phi_{00}(z_k) - \left(\frac{1}{\sqrt{8\pi}^s}\right) \right|^2 + \tag{37}$$

$$+ \sum_{(\mu,\nu)\in I_M^s \times I_M^s;(\mu,\nu)\neq(0,0)} \alpha^{2\mu}\beta^{2\nu} \left| \frac{1}{N}\sum_{k=1}^{N}\Phi_{\mu\nu}(z_k) \right|^2 + \tag{38}$$

$$+\frac{1}{N^2}\sum_{k,l=1}^{N}\sum_{(\mu,\nu)\notin I_M^s \times I_M^s} \Phi_{\mu\nu}(z_k)\overline{\Phi_{\mu\nu}(z_l)}\alpha^{2\mu}\beta^{2\nu}. \tag{39}$$

Due to the uniform distribution of the sequence $\{z_k\}$ and Theorem 1 there is an index $N(\varepsilon, M)$ with the property that

$$\left| \frac{1}{N}\sum_{k=1}^{N}\Phi_{\mu\nu}(z_k) - \int_{\mathbb{C}^s}\Phi_{\mu\nu}(z)w(z)dz \right| < \varepsilon$$

for all $N > N(\varepsilon, M)$ for $(\mu, \nu) \in I_M \times I_M$. So we have the sum of term (37) and (38) less than $A\varepsilon$ where A is an absolute constant. We consider the residue term in the equation above and separate the summation over the points z_k into a sum where $z_k, z_l \in K_R$, a sum where only $z_k \in K_R$ (due to symmetry the same as if $z_l \in K_R$) and into a sum where $z_k, z_l \notin K_R$:

$$\frac{1}{N^2}\sum_{k,l=1}^{N}\sum_{(\mu,\nu)\notin I_M^s \times I_M^s} \Phi_{\mu\nu}(z_k)\overline{\Phi_{\mu\nu}(z_l)}\alpha^{2\mu}\beta^{2\nu} =$$

$$= \frac{1}{N^2} \sum_{k,l=1, z_k, z_l \in K_R}^{N} \sum_{(\mu,\nu) \notin I_M^s \times I_M^s} \Phi_{\mu\nu}(z_k) \overline{\Phi_{\mu\nu}(z_l)} \alpha^{2\mu} \beta^{2\nu} + \quad (40)$$

$$+ \frac{2}{N^2} \sum_{k,l=1, z_k \in K_R, z_l \notin K_R}^{N} \sum_{(\mu,\nu) \notin I_M^s \times I_M^s} \Phi_{\mu\nu}(z_k) \overline{\Phi_{\mu\nu}(z_l)} \alpha^{2\mu} \beta^{2\nu} + \quad (41)$$

$$+ \frac{1}{N^2} \sum_{k,l=1, z_k, z_l \notin K_R}^{N} \sum_{(\mu,\nu) \notin I_M^s \times I_M^s} \Phi_{\mu\nu}(z_k) \overline{\Phi_{\mu\nu}(z_l)} \alpha^{2\mu} \beta^{2\nu}. \quad (42)$$

Due to the assumption on M term (40) is less than ε. Due to remark (3) and the assumption on K_R term (41) and term (42) are less than ε. □

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