A note on the diaphony generated by spherical harmonics

Peter Jez

Technical Report 2018-02 June 2018

Department of Computer Sciences

Jakob-Haringer-Straße 2
5020 Salzburg
Austria
www.cosy.sbg.ac.at

Technical Report Series

A note on the diaphony generated by spherical harmonics

Peter Jez, University of Salzburg,
peter.jez.g@gmail.com

June 10, 2018

Abstract

In QMC integration a measure of the quality of the used point sequence $\{x_k\}_{k\geq 1} \in \mathbb{R}^s$ is the so called Diaphony. The classical Diaphony (see [13]) is defined for sequences in the s-dimensional unit cube. In the following we consider the case of QMC integration over the surface of the s-dimensional unit sphere with respect to Lebesgue measure. A key role in Analysis on the s-dimensional unit sphere is played by the spherical harmonics. In section 1 we recall well known properties of the system of spherical harmonics used in the sequel. In section 2 we introduce a reproducing kernel Hilbert space and compute the kernels for special cases. Section 3 is dedicated to the application of the methods developed before to the uniform distribution of sequences on the unit sphere and give criteria for the uniform distribution. In section 4 we discuss briefly the limit distribution of the diaphony and investigate the approximate computation of the distribution function.

1 Preliminaries

At first we recall basic facts about spherical harmonics. In the following we denote the surface of the s-dimensional unit sphere by S^{s-1}:

$$S^{s-1} = \left\{x = (x_1, \ldots, x_s) \in \mathbb{R}^s | x_1^2 + \ldots + x_s^2 = 1\right\}.$$

An important constant is the Lebesgue measure of the surface of the unit sphere:

$$\omega_s = \int_{S^{s-1}} d\omega = \frac{2\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}. \tag{1}$$

Definition 1. *(see [9]) Let $Q_n(x)$ be a homogenous polynomial of degree n in s dimensions with*

$$\Delta Q_n = 0 \tag{2}$$

with

$$\Delta = \sum_{k=1}^{s} \frac{\partial^2}{\partial x_k^2}. \tag{3}$$

Let $x = \tau\xi$ with $\xi \in S^{s-1}$. Then

$$S_n(\xi) = \frac{1}{\tau^n} Q_n(\tau\xi) = Q_n(\xi) \tag{4}$$

is called a regular spherical harmonic of order n in s dimensions.

The following lemma sums up well known properties of spherical harmonics used in the sequel:

Lemma 1. *(see [9])*

1. *(Orthogonality) Let $d\omega$ denote the Lebesgue measure on the s-dimensional unit sphere. We have*

$$\int_{S^{s-1}} S_n(x) S_m(x) d\omega = 0 \qquad n \neq m \tag{5}$$

2. *The number $N(s,n)$ of linear independent spherical harmonics of degree n in s dimensions is given by $N(s,0) = 1$ and*

$$N(s,n) = \frac{(2n+s-2)\Gamma(n+s-2)}{\Gamma(n+1)\Gamma(s-1)} \tag{6}$$

   for $n \geq 1$ and $s \geq 3$. For $s = 2$ we have $N(s,n) = 2$ for all $n \geq 1$.

3. *There exist $N(s,n)$ linear independent spherical harmonics $S_{n,j}(x)$ of degree n in s dimensions with $1 \leq j \leq N(s,n)$ and every spherical harmonic of degree n in s dimensions can be written as a linear combination of the $S_{n,j}$.*

4. *(Addition theorem of spherical harmonics) Consider the scalar product given by*

$$(f,g) := \int_{S^{s-1}} f(x) g(x) d\omega(x) \tag{7}$$

   where $f(x), g(x)$ are square integrable functions on the unit sphere. Let $S_{n,j}(x)$, $1 \leq j \leq N(s,n)$ be an orthonormal (with respect to the scalar product (7)) set of spherical harmonics of degree n in s dimensions. Then we have

$$\sum_{j=1}^{N(s,n)} S_{n,j}(x) S_{n,j}(y) = \frac{N(s,n)}{\omega_s} P_n(s; \langle x, y \rangle) \tag{8}$$

   where $P_n(s,x)$ denotes the Legendre polynomial of degree n in s dimensions which is given by $P_0(s,x) = 1$ and

$$P_n(s;x) = \left(-\frac{1}{2}\right)^n \frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(n + \frac{s-1}{2}\right)} \left(1-x^2\right)^{\frac{3-s}{2}} \frac{d^n}{dx^n} \left(1-x^2\right)^{n+\frac{s-3}{2}} \tag{9}$$

   for $n \geq 1$ and $\langle .,. \rangle$ denotes the Euclidian scalar product. The Legendre polynomials of degree n in s dimensions are bounded by $|P_n(s,x)| \leq 1$ for

$-1 \leq x \leq 1$. *An alternative representation is Laplaces integral representation:*

$$P_n(s;x) = \frac{\omega_{s-2}}{\omega_{s-1}} \int_{-1}^{1} \left(x + i\sqrt{1-x^2}u\right)^n \left(1-u^2\right)^{\frac{s-4}{2}} du =$$

$$= \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s-2}{2}\right)} \int_{-1}^{1} \left(x + i\sqrt{1-x^2}u\right)^n \left(1-u^2\right)^{\frac{s-4}{2}} du \tag{10}$$

5. *We have*

$$\int_{-1}^{1} P_n(s,t)P_m(s,t)(1-t^2)^{\frac{s-3}{2}} dt = \frac{\omega_s}{\omega_{s-1}} \frac{1}{N(s,n)} \delta_{nm} \tag{11}$$

6. *Let* $x, y \in S^{s-1}$. *We have*

$$\int_{S^{s-1}} P_n(s, \langle x.t \rangle) P_m(s, \langle y.t \rangle) d\omega(t) = 0 \tag{12}$$

for $n \neq m$.

7. *(Funk-Hecke Formula) Let* $x, y \in S^{s-1}$ *and* $f(x) \in C([-1,1])$. *Then we have*

$$\int_{S^{s-1}} f(\langle x.z \rangle) P_n(s, \langle y.z \rangle) d\omega(z) = \lambda P_n(s, x.y) \tag{13}$$

with

$$\lambda = \omega_{s-1} \int_{-1}^{1} f(t) P_n(s,t)(1-t^2)^{\frac{s-3}{2}} dt \tag{14}$$

8. *(Poisson integral) Let* $f(x) \in C\left(S^{s-1}\right)$. *Then we have*

$$f(x) = \lim_{r \to 1^-} \frac{1}{\omega_s} \int_{S^{s-1}} \frac{1-r^2}{\left(1 - r^2 + 2r\langle x, y \rangle\right)^{\frac{s}{2}}} f(y) d\omega(y) \tag{15}$$

uniformly with respect to x *where* $d\omega$ *denotes the Lebesgue measure on the sphere* S^{s-1}.

9. *(Abel Summability) Every* $f(x) \in C\left(S^{s-1}\right)$ *can be approximated uniformly by following expansion:*

$$f(x) = \lim_{r \to 1^-} \sum_{n=0}^{\infty} r^n T_n(s,x) \tag{16}$$

where

$$T_n(s,x) = \sum_{j=1}^{N(s,n)} S_{n,j}(s,x) \int_{S^{s-1}} S_{n,j}(s,y) f(y) d\omega(y) =$$

$$= \frac{N(s,n)}{\omega_s} \int_{S^{s-1}} P_n(s, x.y) f(y) d\omega(y). \tag{17}$$

2 A reproducing kernel Hilbert space spanned by spherical harmonics

In this section we introduce a RKHS based on spherical harmonics. Let $\{\lambda_n\}_{n\geq 0} \in \mathbb{C}$ with

$$\sum_{n=0}^{\infty} N(s,n)\,|\lambda_n|^2 < \infty. \tag{18}$$

We consider the following function space:

$$H := \left\{ f : S^{s-1} \to \mathbb{C}; f(x) = \sum_{n=0}^{\infty} \lambda_n \sum_{j=1}^{N(s,n)} a_{n,j} S_{n,j}(x); \sum_{n=0}^{\infty} \sum_{j=1}^{N(s,n)} |a_{n,j}|^2 < \infty \right\} \tag{19}$$

Equipped with an inner product defined by

$$(\lambda_n S_{n,j}, \lambda_m S_{m,k}) = \delta_{nm}\delta_{jk} \tag{20}$$

H forms a RKHS with kernel $K(x,y) : S^{s-1} \times S^{s-1} \to \mathbb{C}$ given by

$$K(x,y) = \sum_{n=0}^{\infty} |\lambda_n|^2 \sum_{j=1}^{N(s,n)} S_{n,j}(x) S_{n,j}(y) = \frac{1}{\omega_s} \sum_{n=0}^{\infty} |\lambda_n|^2 N(s,n) P_n(s, x.y). \tag{21}$$

For special sequences λ_n this kernel has a closed form:

Example 1. *Let $r \in [0,1)$ and $\lambda_n = r^n$. Then the kernel $K(x,y)$ is given by*

$$K_r(x,y) = \frac{1}{\omega_s} \frac{1 - r^2}{(1 - 2r\langle x,y\rangle + r^2)^{\frac{s}{2}}} \tag{22}$$

where $\langle .,. \rangle$ denotes the Euclidian scalar product on $\mathbb{R}^s$.

Example 2. *Let $s = 3$. The Legendre polynomial of degree n in three dimensions is the normal Legendre polynomial $P_n(x)$ (see [7]). Rainville [11] has proven the following expansion:*

$$\exp(t\cos\theta)\, J_0(t\sin\theta) = \sum_{n=0}^{\infty} P_n(\cos\theta) \frac{t^n}{n!} \tag{23}$$

for $t \in \mathbb{R}$. J_0 denotes the Bessel function of first kind of order 0. Let

$$|\lambda_n|^2 = \frac{4\pi}{n!(2n+1)} \tag{24}$$

in (22). Then we get

$$K(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(\langle x,y\rangle) = \exp(\cos\theta) J_0(\sin\theta) \tag{25}$$

with $\langle x, y\rangle = \cos\theta$*. Let us consider the case* $s > 3$*: Laplaces representation of the Legendre polynomials in* s *dimensions gives an expansion analog to Rainvilles expansion (23):*

$$\sum_{n=0}^{\infty} \frac{P_n(s;t)}{n!} v^n =$$

$$= \sum_{n=0}^{\infty} \frac{v^n}{n!} \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s-2}{2}\right)} \int_{-1}^{1} \left(t + i\sqrt{1-t^2}u\right)^n \left(1-u^2\right)^{\frac{s-4}{2}} du.$$

Exchange of integration and summation is allowed and therefor we get

$$\sum_{n=0}^{\infty} \frac{P_n(s;t)}{n!} v^n = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s-2}{2}\right)} \exp(tv) \int_{-1}^{1} \exp\left(iv\sqrt{1-t^2}u\right) \left(1-u^2\right)^{\frac{s-4}{2}} du. \tag{26}$$

In the integral we set $u = \cos\varphi$ *and we get*

$$\int_{-1}^{1} \exp\left(iv\sqrt{1-t^2}u\right) \left(1-u^2\right)^{\frac{s-4}{2}} du = \int_{0}^{\pi} \exp\left(iv\sqrt{1-t^2}\cos\varphi\right) (\sin\varphi)^{s-3} d\varphi.$$

Due to the formula (see [5])

$$\int_0^{\pi} \exp\left(z\cos\varphi\right) (\sin\varphi)^{2\nu} d\varphi = \frac{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}{\left(\frac{z}{2}\right)^{\nu}} I_\nu(z)$$

we get

$$\int_0^{\pi} \exp\left(iv\sqrt{1-t^2}\cos\varphi\right) (\sin\varphi)^{s-3} d\varphi = \frac{\sqrt{\pi}\Gamma\left(\frac{s-2}{2}\right)}{\left(\frac{iv\sqrt{1-t^2}}{2}\right)^{\frac{s-3}{2}}} I_{\frac{s-3}{2}}\left(iv\sqrt{1-t^2}\right).$$

Due to

$$\frac{I_\nu(ix)}{\left(\frac{ix}{2}\right)^{\nu}} = \frac{J_\nu(x)}{\left(\frac{x}{2}\right)^{\nu}}$$

we have finally

$$\int_0^{\pi} \exp\left(iv\sqrt{1-t^2}\cos\varphi\right) (\sin\varphi)^{s-3} d\varphi = \frac{\sqrt{\pi}\Gamma\left(\frac{s-2}{2}\right)}{\left(\frac{v\sqrt{1-t^2}}{2}\right)^{\frac{s-3}{2}}} J_{\frac{s-3}{2}}\left(v\sqrt{1-t^2}\right).$$

Inserting into (26) we get

$$\sum_{n=0}^{\infty} \frac{P_n(s;t)}{n!} v^n = \frac{\Gamma\left(\frac{s-1}{2}\right)\exp(tv)}{\left(\frac{v\sqrt{1-t^2}}{2}\right)^{\frac{s-3}{2}}} J_{\frac{s-3}{2}}\left(v\sqrt{1-t^2}\right).$$

Setting $t = \cos\theta$ *we achieve*

$$\sum_{n=0}^{\infty} \frac{P_n(s;\cos\theta)}{n!} v^n = \frac{\Gamma\left(\frac{s-1}{2}\right)\exp(v\cos\theta)}{\left(\frac{v\sin\theta}{2}\right)^{\frac{s-3}{2}}} J_{\frac{s-3}{2}}\left(v\sin\theta\right). \tag{27}$$

Now set

$$|\lambda_n|^2 = \frac{\omega_s}{n!N(s,n)}$$

in formula 21. With these coefficients and setting $v = 1$ *in (27) we get the kernel*

$$K(x,y) = \sum_{n=0}^{\infty} |\lambda_n|^2 \sum_{j=1}^{N(s,n)} S_{n,j}(x) S_{n,j}(y) =$$

$$= \sum_{n=0}^{\infty} \frac{P_n(s;\cos\theta)}{n!} = \frac{\Gamma\left(\frac{s-1}{2}\right)\exp(\cos\theta)}{\left(\frac{\sin\theta}{2}\right)^{\frac{s-3}{2}}} J_{\frac{s-3}{2}}(\sin\theta)\,. \tag{28}$$

For s *even this kernel can be expressed in elementary functions.*

Remark 1. *The generating function (27) shows the relation between the Legendre polynomials in* s *dimensions* $P_n(s;\cos\theta)$ *and the ultraspherical polynomials* $C_n^{(\lambda)}(\cos\theta)$ *(see [5]): The generating function of the ultraspherical polynomials is given by*

$$\sum_{n=0}^{\infty} \frac{C_n^{(\lambda)}(\cos\theta)}{(2\lambda)_n} z^n = \Gamma\left(\lambda + \frac{1}{2}\right) exp(z\cos\theta) \frac{J_{\lambda-\frac{1}{2}}(z\sin\theta)}{\left(\frac{z\sin\theta}{2}\right)^{\lambda-\frac{1}{2}}}.$$

From this we have

$$P_n(s;\cos\theta) = n! \frac{C_n^{\left(\frac{s-2}{2}\right)}(\cos\theta)}{(s-2)_n}.$$

Now let $f(x) \in H$. Consider a point sequence $\{x_k\}_{k\geq 1} \in S^{s-1}$ and the MC integration error given by

$$E_N(f) = \left| \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \frac{1}{\omega_s} \int_{S^{s-1}} f(x) d\omega_s \right|$$

where $d\omega$ denotes the Lebesgue measure on the unit sphere S^{s-1}. By RKHS property we get

$$E_N(f) = \left| \frac{1}{N} \sum_{k=1}^{N} (f, K(.,x_k)) - a_{0,0}\lambda_0 \right| =$$

$$= \left| \left(f, \frac{1}{N} \sum_{k=1}^{N} K(.,x_k) - |\lambda_0|^2 S_{0,0} \right) \right| \leq$$

$$\|f\|. \left(\frac{1}{N^2} \sum_{n,k=1} K(x_k, x_l) - \frac{|\lambda_0|^2}{\omega_s} \right)^{\frac{1}{2}}.$$

where $\|.\|$ denotes the norm in H. The term

$$D_N(x_k) := \left(\frac{1}{N^2} \sum_{n,k=1} K(x_k, x_l) - \frac{|\lambda_0|^2}{\omega_s} \right)^{\frac{1}{2}} \tag{29}$$

is called the Diaphony $D_N(x_k)$ of the sequence $\{x_k\}_{k\geq 1} \in S^{s-1}$ (see [14]).

Remark 2. *A similar error bound is found in [10]. It is computed by the estimation of the integration error for functions from a weighted Sobolew space.*

An immedeate consequence is the following

Corollary 1. *We have $E_N(f) \to 0$ for all $f \in H$ if we have $D_N(x_k) \to 0$ for $N \to \infty$.*

3 Uniform distribution of point sequences $\{x_k\}_{k\geq 1} \in S^{s-1}$

In this section we consider MC integration over the unit sphere for a more general class of functions. Let ω be the Lebesgue measure on the s-dimensional unit sphere and $\{x_k\} \in S^{s-1}$. We consider the integration error

$$E_N(f) = \left| \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \frac{1}{\omega_s} \int_{S^{s-1}} f(x) d\omega \right| \tag{30}$$

for $f \in C\left(S^{s-1}\right)$. We start with the following

Definition 2. *A sequence of points $\{x_k\}_{k\geq 1} \in S^{s-1}$ is uniform distributed if we have $E_N(f) \to 0$ for $N \to \infty$ for all $f \in C\left(S^{s-1}\right)$.*

One algorithm get sequences suitable for numerical integration is to lift a low discrepancy sequence on $[0,1)^{s-1}$ onto the unit sphere S^{s-1}. Examples for such sequences on S^2 (lifted (t,m,s)-nets by an area preserving transformation) and numerical results are given in [1]. Lifting algorithms are also found in [4]. Our aim is to show

Theorem 1. *A sequence of points $\{x_k\}_{k\geq 1} \in S^{s-1}$ is uniform distributed if and only if we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} S_{n,j}(x_k) = 0 \tag{31}$$

for all $n \geq 1$ and all $j = 1, \ldots, N(s,n)$.

We start with the following

Lemma 2. *Let* $0 < r < 1$, $\{x_k\}_{k\geq 1} \in S^{s-1}$ *and*

$$K_r : S^{s-1} \times S^{s-1} \to \mathbb{R} : K_r(x,y) = \frac{1}{\omega_s} \frac{1-r^2}{(1-2r\langle x.y\rangle + r^2)^{\frac{s}{2}}}$$

Let

$$D_{N,r}(x_k) := \left(\frac{1}{N^2} \sum_{n,k=1}^{N} K_r(x_k, x_l) - \frac{1}{\omega_s} \right)^{\frac{1}{2}}. \tag{32}$$

The sequence $\{x_k\}_{k\geq 1} \in S^{s-1}$ *is uniform distributed if and only if we have*

$$\lim_{N\to\infty} D_{N,r}(x_k) = 0$$

for all $0 < r < 1$.

Proof. Let $f(x) \in C(S^{s-1})$ and $d\omega$ be the Lebesgue measure on S^{s-1}. We define

$$f_r(x) := \frac{1}{\omega_s} \int_{S^{s-1}} \frac{1-r^2}{(1+r^2-2r\langle x.y\rangle)^{\frac{s}{2}}} f(y) d\omega(y)$$

Then we have

$$E_N(f) \leq \left| \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \frac{1}{N} \sum_{k=1}^{N} f_r(x_k) \right| +$$

$$+ \left| \frac{1}{N} \sum_{k=1}^{N} f_r(x_k) - \frac{1}{\omega_s} \int_{S^{s-1}} f_r(x) d\omega(x) \right| +$$

$$+ \left| \frac{1}{\omega_s} \int_{S^{s-1}} f_r(x) d\omega(x) - \frac{1}{\omega_s} \int_{S^{s-1}} f(x) d\omega(x) \right|. \tag{33}$$

We start with the first and the last summand in (33): Due to the uniform convergence with respect to x for all $\varepsilon > 0$ there is an $R_0(\varepsilon)$ with the fact that

$$|f(x) - f_r(x)| < \varepsilon$$

for all $x \in S^{s-1}$ for all r with $R_0(\varepsilon) < r < 1$. So we get

$$E_N(f) \leq 2\varepsilon + \left| \frac{1}{N} \sum_{k=1}^{N} f_r(x_k) - \frac{1}{\omega_s} \int_{S^{s-1}} f_r(x) d\omega(x) \right|$$

for $R_0(\varepsilon) < r < 1$. We fix an r with $R_0(\varepsilon) < r < 1$. The remaining term can be estimated in the following way:

$$\left| \frac{1}{N} \sum_{k=1}^{N} f_r(x_k) - \frac{1}{\omega_s} \int_{S^{s-1}} f_r(x) d\omega(x) \right| =$$

$$= \frac{1}{\omega_s} \left| \int_{S^{s-1}} \left(\frac{1}{N} \sum_{k=1}^{N} \frac{1-r^2}{1+r^2-2r\langle x_k.y\rangle)^{\frac{s}{2}}} - 1 \right) f(y) d\omega(y) \right| \leq$$

$$\leq \frac{1}{\omega_s} \left(\int_{S^{s-1}} \left(\frac{1}{N} \sum_{k=1}^{N} \frac{1-r^2}{(1+r^2-2r\langle x_k.y\rangle)^{\frac{s}{2}}} - 1 \right)^2 d\omega(y) \right)^{\frac{1}{2}} \cdot \left(\int_{S^{s-1}} |f|^2 d\omega \right)^{\frac{1}{2}}$$

due to the representation of $f_r(x)$, exchanging the integration sequence in the integral over x, the fact

$$\frac{1}{\omega_s} \int_{S^{s-1}} \frac{1-r^2}{1+r^2-2r\langle x.y\rangle)^{\frac{s}{2}}} d\omega(y) = 1$$

and Cauchy-Schwarz inequality. The factor independent from f can be computed:

$$\int_{S^{s-1}} \left(\frac{1}{N} \sum_{k=1}^{N} \frac{1-r^2}{(1+r^2-2r\langle x_k.y\rangle)^{\frac{s}{2}}} - 1 \right)^2 d\omega(y) =$$

$$\frac{1}{N^2} \int_{S^{s-1}} \sum_{k,l=1}^{N} \left(\frac{1-r^2}{(1+r^2-2r\langle x_k.y\rangle)^{\frac{s}{2}}} - 1 \right) \left(\frac{1-r^2}{(1+r^2-2r\langle x_l.y\rangle)^{\frac{s}{2}}} - 1 \right) d\omega(y) =$$

$$= \frac{1}{N^2} \int_{S^{s-1}} \sum_{k,l=1}^{N} \sum_{n,m=1}^{\infty} N(s,n) r^n P_n(s, x_k.y) N(s,m) r^m P_m(s, x_l.y) d\omega(y) =$$

$$= \frac{1}{N^2} \int_{S^{s-1}} \sum_{n,m=1}^{\infty} \sum_{k,l=1}^{N} N(s,n) r^n P_n(s, x_k.y) N(s,m) r^m P_m(s, x_l.y) d\omega(y).$$

Due to the uniform convergence of the series expansion integration and summation can be exchanged and therefor we have

$$\int_{S^{s-1}} \sum_{n,m=1}^{\infty} \sum_{k,l=1}^{N} N(s,n) r^n P_n(s, x_k.y) N(s,m) r^m P_m(s, x_l.y) d\omega(y) =$$

$$= \sum_{k,l=1}^{N} \sum_{n,m=1}^{\infty} \int_{S^{s-1}} N(s,n) r^n P_n(s, x_k.y) N(s,m) r^m P_m(s, x_l.y) d\omega(y).$$

By Funk-Hecke formula and (11) we can compute the integral:

$$\int_{S^{s-1}} N(s,n) r^n P_n(s, x_k.y) N(s,m) r^m P_m(s, x_l.y) d\omega(y) =$$

$$r^{2n} N(s,n) N(s,n) \frac{\omega_s}{N(s,n)} P_n(s, x_k.x_l) \delta_{nm}$$

and therefor

$$\frac{1}{N^2} \sum_{k,l=1}^{N} \sum_{n,m=1}^{\infty} \int_{S^{s-1}} N(s,n) r^n P_n(s, x_k.y) N(s,m) r^m P_m(s, x_l.y) d\omega(y) =$$

$$= \omega_s \frac{1}{N^2} \sum_{k,l=1}^{N} \sum_{n=1}^{\infty} r^{n+m} N(s,n) P_n(s, x_k.x_l) =$$

$$= \omega_s \frac{1}{N^2} \left(\sum_{k,l=1}^{N} \frac{1-r^4}{(1+r^4-2r^2\langle x_k.x_l\rangle)^{\frac{s}{2}}} - 1 \right) =: \omega_s^2 \left(D_{N,r^2}(x_k)\right)^2 .$$

So we have the estimation

$$\left| \frac{1}{N} \sum_{k=1}^{N} f_r(x_k) - \frac{1}{\omega_s} \int_{S^{s-1}} f_r(x) d\omega(x) \right| \leq \omega_s D_{N,r^2}(x_k) \left(\int_{S^{s-1}} |f|^2 d\omega \right)^{\frac{1}{2}}$$

and finally

$$E_N(f) \leq 2\varepsilon + \omega_s D_{N,r^2}(x_k) \left(\int_{S^{s-1}} |f|^2 d\omega \right)^{\frac{1}{2}} .$$

So if we have $D_{N,r}(x_k) \to 0$ for all $0 < r < 1$ we have $E_N(f) \to 0$. Now suppose that $\{x_k\}$ is uniformly distributed on S^{s-1}. We have

$$(D_{N,r}(x_k))^2 = \sum_{n=1}^{\infty} r^n \sum_{j=1}^{N(s,n)} \left(\frac{1}{N} \sum_{k=1}^{N} S_{n,j}(x_k) \right)^2 .$$

By addition theorem, the orthonormality of $S_{n,j}$ and the boundedness of the Legendre polynomials in s dimensions we have $(D_{N,r}(x_k))^2 \to 0$. □

We continue with the proof Theorem 1:

Proof. If $\{x_k\}_{k\geq 1} \in S^{s-1}$ is uniformly distributed in the sense of Definition 2 we have fulfilled (31) because the spherical harmonics are continous functions on the sphere. According to the previous lemma to show the suffuciency it is enough to show that (31) implies the assumption of the lemma. Let $\varepsilon > 0$ be given and $0 < r < 1$ be given. Due to $|P_n(s,t)| \leq 1$ for $-1 \leq t \leq 1$ and

$$\left| \sum_{n=1}^{\infty} N(s,n) r^n P_n(s,t) \right| \leq \sum_{n=1}^{\infty} N(s,n) r^n = \frac{1+r}{(1-r)^{s-1}}$$

we find an index $M_0(\varepsilon)$ with

$$\left| \sum_{n=M}^{\infty} N(s,n) r^n P_n(s,t) \right| < \varepsilon$$

for all $M > M_0(\varepsilon)$. So we have

$$\left| \frac{1}{N^2} \sum_{n,k=1}^{N} K_r(x_k, x_l) - \frac{1}{\omega_s} \right| = \frac{1}{\omega_s} \left| \frac{1}{N^2} \sum_{k,l=1}^{N} \sum_{n=1}^{\infty} N(s,n) r^n P_n(s, x_k.x_l) \right| <$$

$$< \frac{1}{\omega_s} \left| \frac{1}{N^2} \sum_{k,l=1}^{N} \sum_{n=1}^{M} N(s,n) r^n P_n(s, x_k.x_l) \right| + \frac{1}{\omega_s} \varepsilon =$$

$$= \left| \frac{1}{N^2} \sum_{n=1}^{M} r^n \sum_{j=1}^{N(s,n)} \sum_{k,l=1}^{N} S_{n,j}(x_k) S_{n,j}(x_l) \right| + \frac{1}{\omega_s} \varepsilon \leq$$

$$\leq \sum_{n=1}^{M} r^n \sum_{j=1}^{N(s,n)} \left| \frac{1}{N} \sum_{k=1}^{N} S_{n,j}(x_k) \right|^2 + \frac{1}{\omega_s} \varepsilon.$$

Due to the assumption of the theorem we can find an index $N(\varepsilon)$ with

$$\left| \frac{1}{N} \sum_{k=1}^{N} S_{n,j}(x_k) \right| < \varepsilon$$

for $N > N(\varepsilon)$ for all $1 \leq n \leq M$ and all $1 \leq j \leq N(s,n)$. So we have

$$\left| \frac{1}{N^2} \sum_{n,k=1}^{N} K_r(x_k, x_l) - \frac{1}{\omega_s} \right| < \frac{1+r}{(1-r)^{s-1}} \varepsilon^2 + \frac{1}{\omega_s} \varepsilon.$$

Now the proof of the theorem is complete. □

A closer look to the previous results gives another criterion for uniform distribution on the unit sphere:

Corollary 2. *The sequence* $\{x_k\}_{k \geq 1} \in S^{s-1}$ *is uniformly distributed if and only if we have*

$$\lim_{N \to \infty} D_{N,r_0}(x_k) = 0 \tag{34}$$

for one $0 < r_0 < 1$.

Proof. According to Theorem 1 we have to show that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} S_{n,j}(x_k) = 0$$

for all $n \geq 1$ and $1 \leq j \leq N(s,n)$ if condition 34 is fulfilled. We consider

$$\left(\frac{1}{N} \sum_{k=1}^{N} S_{n,j}(x_k) \right)^2$$

for an arbitrary n, j and we have the estimation:

$$\left(\frac{1}{N} \sum_{k=1}^{N} S_{n,j}(x_k) \right)^2 = \frac{1}{r_0^n} r_0^n \left(\frac{1}{N} \sum_{k=1}^{N} S_{n,j}(x_k) \right)^2 \leq$$

$$\leq \frac{1}{r_0^n} \sum_{m=1}^{\infty} r_0^m \sum_{j=1}^{N(s,n)} \left(\frac{1}{N} \sum_{k=1}^{N} S_{n,j}(x_k) \right)^2 = \frac{1}{r_0^n} D_{N,r_0}^2(x_k) \to 0.$$

The proof that uniform distribution implies $D_{N,r_0} \to 0$ is a one to one copy of the proof of theorem 1. $\square$

4 The limit distribution of the Diaphony

Limit distributions of Discrepancies in the unit cube are invesitigated in [2], where also Berry-Esseen bounds are shown. We are interested in the distribution of the expression

$$N \left(D_N(x_k) \right)^2 = \sum_{n=1}^{\infty} |\lambda_n|^2 \sum_{j=1}^{N(s,n)} \left(\frac{1}{\sqrt{N}} \sum_{k=1}^{N} S_{n,j}(x_k) \right)^2$$

for $\{x_k\}_{k \geq 1} \in S^{s-1}$ independent identically distributed $U\left(S^{s-1}\right)$ random variables for N very large. We consider the approximant

$$N \left(D_N^{(M)}(x_k) \right)^2 := \sum_{n=1}^{M} |\lambda_n|^2 \sum_{j=1}^{N(s,n)} \left(\frac{1}{N} \sum_{k=1}^{N} S_{n,j}(x_k) \right)^2$$

By multivariate CLT we have

$$Y_M(x) := \frac{1}{\sqrt{N}} \sum_{k=1}^{M} \begin{pmatrix} S_{1,1}(x_k) \\ \vdots \\ S_{1,N(s,1)}(x_k) \\ \vdots \\ S_{M,N(s,M)}(x_k) \end{pmatrix} \to_d N\left(0, \Sigma\right) \tag{35}$$

where Σ is the covariance matrix of the vector

$$\begin{pmatrix} S_{1,1}(x) \\ \vdots \\ S_{1,N(s,1)}(x) \\ \vdots \\ S_{M,N(s,M)}(x) \end{pmatrix} \tag{36}$$

for $x \sim U\left(S^{s-1}\right)$. From the orthonormality of the spherical harmonics we obtain

$$\Sigma = (\Sigma)_{ij} = \frac{1}{\omega_s} \delta_{ij}$$

By continuity theorem (see [3]) we obtain (due to (18))

$$N \left(D_N(x_k) \right)^2 \to_d \frac{1}{\omega_s} \sum_{k=1}^{\infty} |\lambda_k|^2 \sum_{j=1}^{N(s,k)} X_{k,j}^2 \tag{37}$$

with $X_{k,j} \sim N(0,1)$ i.i.d. random variables which can be expressed by χ^2 distribution:

$$N\left(D_N(x_k)\right)^2 \to_d \frac{1}{\omega_s}\sum_{k=1}^{\infty}|\lambda_k|^2\,\chi^2\left(N(s,k)\right) \tag{38}$$

4.1 Computation of the moments of the limit distribution

In the following we want to compute the moments of the distribution (38). From multinomial theorem we obtain the following relation:

$$\left(\sum_{n=1}^{\infty}a_n\right)^k = \sum\binom{k}{k_1,k_2,\ldots,k_{|p(k)|}}\sum_{n_1,\ldots,n_{|p(k)|}=1;n_i\neq n_j}^{\infty}a_{n_1}^{k_1}\ldots a_{n_{|p(k)|}}^{k_{|p(k)|}}. \tag{39}$$

where the outer sum is done over all partitions $p(k)$ of k with elements $k_1,k_2,\ldots,k_{|p(k)|}$. Now we consider the random variable

$$Y_s := \frac{1}{\omega_s}\sum_{k=1}^{\infty}|\lambda_k|^2\chi^2\left(N(s,k)\right)$$

with independent χ^2 random variables. As in the previous section we assume

$$\sum_{k=1}^{\infty}|\lambda_k|^2N(s,k) < \infty.$$

From (39) we get

$$Y_s^k = \frac{1}{\omega_s^k}\sum_{p(k) partition of k}\binom{k}{k_1,k_2,\ldots,k_{|p(k)|}}\times$$

$$\sum_{n_1,\ldots,n_{|p(k)|}=1;n_i\neq n_j}^{\infty}|\lambda_{n_1}|^{2k_1}\ldots|\lambda_{n_{|p(k)|}}|^{2k_{|p(k)|}}\times$$

$$\left(\chi^2\left(N(s,n_1)\right)\right)^{k_1}\ldots\left(\chi^2\left(N(s,n_{|p(k)|})\right)\right)^{k_{|p(k)|}}.$$

The moments of $\chi^2(m)$ distributions are given by

$$E\left(\chi^2(m)\right)^l = 2^l\left(\frac{m}{2}-1\right)_l$$

and by the independence of the χ^2 random variables we get

$$\mu_k^{(s)} := EY_s^k = \frac{2^k}{\omega_s^k}\sum_{p(k) partition of k}\binom{k}{k_1,k_2,\ldots,k_{|p(k)|}}\times$$

$$\sum_{n_1,\ldots,n_{|p(k)|}=1;n_i\neq n_j}^{\infty}|\lambda_{n_1}|^{2k_1}\ldots|\lambda_{n_{|p(k)|}}|^{2k_{|p(k)|}}\times$$

$$\left(\frac{N(s,n_1)}{2}-1\right)_{k_1}\cdots\left(\frac{N(s,n_{|p(k)|})}{2}-1\right)_{k_{|p(k)|}}$$

In the following we will have a closer look the distribution (38) for the particular sequence $\lambda_k = r^k$ for $0 < r < 1$ and small values of s. We investigate the computation of the moments by much more elementary tools. By simple calculation we obtain

Lemma 3. *The characteristic function $\varphi_s(t)$ of the right hand side of (38) is given by*

$$\varphi_s(t) = \frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{2itr^{2n}}{\omega_s}\right)^{\frac{N(s,n)}{2}}} \tag{40}$$

The most simple case is $s = 2$: We have $N(2, n) = 2$ and therefor

$$\varphi_2(t) = \frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{itr^{2n}}{\pi}\right)} = \frac{1}{\left(\frac{itr^2}{\pi};r^2\right)_\infty}.$$

By the q-exponential (see [5])

$$e_q(x) = \frac{1}{((1-q)x;q)_\infty} = \sum_{n=0}^{\infty}\frac{(1-q)^n x^n}{(q;q)_n}$$

we have the representation

$$\varphi_2(t) = e_{r^2}\left(\frac{itr^2}{\pi(1-r^2)}\right)$$

and therefor the expansion

$$\varphi_2(t) = \sum_{n=0}^{\infty}\frac{1}{(r^2;r^2)_n}\left(\frac{ir^2}{\pi}\right)^n t^n.$$

This gives the moments immedeately:

$$\mu_n^{(2)} = n!\frac{1}{(r^2;r^2)_n}\left(\frac{r^2}{\pi}\right)^n.$$

For $s = 3$ we have $N(3, n) = 2n + 1$ and $\omega_3 = 4\pi$ and therefor

$$\varphi_3(t) = \frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{itr^{2n}}{2\pi}\right)^{n+\frac{1}{2}}}$$

The characteristic function fulfills the following functional equation:

$$\varphi_3(t) = \frac{1}{\left(1-\frac{itr^2}{2\pi}\right)^{\frac{1}{2}}}e_{r^2}\left(\frac{itr^2}{2\pi(1-r^2)}\right)\varphi_3\left(r^2 t\right)$$

From the expansion of the q-exponential and the expansion

$$\frac{1}{\left(1-\frac{itr^2}{2\pi}\right)^{\frac{1}{2}}} = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} \left(\frac{ir^2 t}{2\pi}\right)^k$$

and the value $\mu_0^{(3)} = 1$ we get the recursion formula

$$\left(\frac{2\pi}{r^2}\right)^n \frac{\mu_n^{(3)}}{n!} = \sum_{k=0}^{n} (2\pi)^{n-k} \frac{\mu_{n-k}^{(3)}}{(n-k)!} \sum_{l=0}^{k} \frac{1}{(r^2;r^2)_{k-l}} \frac{(2l)!}{2^{2l}(l!)^2}$$

for the moments $\mu_n^{(3)}$.

4.2 Approximate computation of the limit distribution

It is known that the distribution function of the weigthed sum of M independent χ^2 random variables can be expressed by Srivastavas generalized hypergeometric function. So it would be interesting to approximate the distribution of Y_s by

$$F_M(x) = P(Y_{s,M} \leq z)$$

with

$$Y_{s,M} := \frac{1}{\omega_s} \sum_{n=1}^{M} |\lambda_n|^2 \chi^2\left(N(s,n)\right)$$

The dimension s is a fixed value and therefor omitted in the notation of the distribution functions. An important assumption is the integrability of Y_s:

$$E(Y_s) = E\left(\frac{1}{\omega_s} \sum_{n=1}^{\infty} |\lambda_n|^2 \chi^2(N(s,n))\right) = \frac{1}{\omega_s} \sum_{n=1}^{\infty} |\lambda_n|^2 N(s,n) < \infty. \tag{41}$$

Due to (41) there is an index M_a with the property that

$$\sum_{n=M+1}^{\infty} |\lambda_n|^2 < \frac{\omega_s^2}{8}$$

for all $M > M_a$ and there is an index M_b with the property that

$$\sum_{n=M+1}^{\infty} |\lambda_n|^2 N(s,n) < 1$$

for all $M > M_b$. Now let $0 < \alpha < \frac{1}{3}$ be given. Then there is an index $M_c(\alpha)$ with the property that

$$\left(\sum_{n=M+1}^{\infty} |\lambda_n|^2 N(s,n)\right)^{1-\alpha} < \omega_s \frac{\pi}{2} \tag{42}$$

for all $M > M_c(\alpha)$. There is also an index $M_d(\alpha)$ with the property that

$$\left(\sum_{n=M+1}^{\infty} N(s,n)\,|\lambda_n|^2\right)^{1-2\alpha} < 1 \tag{43}$$

for all $M > M_d(\alpha)$.

Let $M^*(\alpha) = \max\left(M_a, M_b, M_c(\alpha), M_d(\alpha)\right)$.

Theorem 2. *Let $s \geq 3$ and $0 < \alpha < \frac{1}{3}$ be given. Let $M > M^*(\alpha)$. We have*

$$|F(x) - F_M(x)| \leq \frac{8}{\pi(s-2)}\left(\frac{\omega_s}{2|\lambda_1|^2}\right)^{\frac{s}{2}}\left(\sum_{n=M+1}^{\infty} N(s,n)\,|\lambda_n|^2\right)^{\frac{\alpha s}{2}} +$$

$$\frac{2x}{\omega_s^2\pi}\left(\sum_{n=M+1}^{\infty} N(s,n)\,|\lambda_n|^2\right)^{1-3\alpha} +$$

$$\frac{3x}{4\omega_s\pi}\left(\sum_{n=M+1}^{\infty} N(s,n)|\lambda_n|^2\right)^{1-2\alpha} \tag{44}$$

for $x > 0$.

Proof. The characteristic function $\varphi(t)$ of $F(x)$ is given by (see Lemma 40)

$$\varphi(t) = \frac{1}{\prod_{n=1}^{\infty}\left(1 - \frac{2it|\lambda_n|^2}{\omega_s}\right)^{\frac{N(s,n)}{2}}}. \tag{45}$$

To get an integral representation we do the same computations as [6]. The distribution function is given by the inversion theorem:

$$F(b) - F(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(t)\frac{e^{-iat} - e^{-ibt}}{it}dt.$$

Using $F(0) = 0$ and computing the complex exponential function (principal value) we get

$$F(x) = \frac{2}{\pi}\int_0^{\infty}\cos\left(\frac{xt}{2} - \frac{1}{2}\sum_{n=1}^{\infty} N(s,n)\arctan\frac{2t|\lambda_n|^2}{\omega_s}\right)\frac{\sin\frac{xt}{2}}{t}\frac{dt}{\prod_{n=1}^{\infty}\left(1 + \frac{4t^2|\lambda_n|^4}{\omega_s^2}\right)^{\frac{N(s,n)}{4}}}$$

$F_M(x)$ can be expressed by its characteristic function:

$$F_M(x) = \frac{2}{\pi}\int_0^{\infty}\cos\left(\frac{xt}{2} - \frac{1}{2}\sum_{n=1}^{M} N(s,n)\arctan\frac{2t|\lambda_n|^2}{\omega_s}\right)\frac{\sin\frac{xt}{2}}{t}\frac{dt}{\prod_{n=1}^{M}\left(1 + \frac{4t^2|\lambda_n|^4}{\omega_s^2}\right)^{\frac{N(s,n)}{4}}}.$$

Now let

$$K := \frac{1}{\left(\sum_{n=M+1}^{\infty} N(s,n)\,|\lambda_n|^2\right)^{\alpha}}$$

for $M > M^*(\alpha)$ and define

$$F^{(K)}(x) := \frac{2}{\pi}\int_0^K \cos\left(\frac{xt}{2} - \frac{1}{2}\sum_{n=1}^{\infty} N(s,n)\arctan\frac{2t|\lambda_n|^2}{\omega_s}\right)\frac{\sin\frac{xt}{2}}{t}\frac{dt}{\prod_{n=1}^{\infty}\left(1+\frac{4t^2|\lambda_n|^4}{\omega_s^2}\right)^{\frac{N(s,n)}{4}}}$$

and

$$F_M^{(K)}(x) = \frac{2}{\pi}\int_0^K \cos\left(\frac{xt}{2} - \frac{1}{2}\sum_{n=1}^{M} N(s,n)\arctan\frac{2t|\lambda_n|^2}{\omega_s}\right)\frac{\sin\frac{xt}{2}}{t}\frac{dt}{\prod_{n=1}^{M}\left(1+\frac{4t^2|\lambda_n|^4}{\omega_s^2}\right)^{\frac{N(s,n)}{4}}}.$$

We get the estimation

$$|F(x) - F_M(x)| \leq \left|F(x) - F^{(K)}(x)\right| + \left|F_M^{(K)} - F_M(x)\right| +$$

$$+\left|F^{(K)}(x) - F_M^{(K)}(x)\right|.$$

The terms $\left|F(x) - F^{(K)}(x)\right|$ and $\left|F_M^{(K)} - F_M(x)\right|$ can be handeled in the same way. Investigation is done for the first one and can be repeated one to one for the second one:

$$\left|F(x) - F^{(K)}(x)\right| \leq \frac{2}{K\pi}\int_K^{\infty}\frac{dt}{\prod_{n=1}^{M}\left(1+\frac{4t^2|\lambda_n|^4}{\omega_s^2}\right)^{\frac{N(s,n)}{4}}} \leq$$

$$\leq \frac{2}{\pi}\frac{C\left(\omega_s, |\lambda_1|^2, |\lambda_2|^2, |\lambda_3|^2\right)}{K^{\frac{N(s,1)}{2}+\frac{N(s,2)}{2}+\frac{N(s,3)}{2}}}.$$

For $M \geq 3$ we have the estimation

$$\left|F(x) - F^{(K)}(x)\right| + \left|F_M^{(K)} - F_M(x)\right| \leq \frac{4}{\pi}\frac{C\left(\omega_s, |\lambda_1|^2, |\lambda_2|^2, |\lambda_3|^2\right)}{K^{\frac{N(s,1)}{2}+\frac{N(s,2)}{2}+\frac{N(s,3)}{2}}}.$$

For larger dimensions it is not necessary to consider 3 factors. Because of $N(s,1) = s$ for $s \geq 3$ we have the estimation

$$\left|F(x) - F^{(K)}(x)\right| \leq \frac{4}{\pi(s-2)}\left(\frac{\omega_s}{2|\lambda_1|^2}\right)^{\frac{s}{2}}\frac{1}{K^{\frac{s}{2}}} =$$

$$\frac{4}{\pi(s-2)}\left(\frac{\omega_s}{2|\lambda_1|^2}\right)^{\frac{s}{2}}\left(\sum_{n=M+1}^{\infty} N(s,n)\,|\lambda_n|^2\right)^{\frac{\alpha s}{2}}$$

resp.

$$\left|F(x)-F^{(K)}(x)\right|+\left|F_M^{(K)}(x)-F_M(x)\right|\le$$

$$\frac{8}{\pi(s-2)}\left(\frac{\omega_s}{2|\lambda_1|^2}\right)^{\frac{s}{2}}\left(\sum_{n=M+1}^{\infty}N(s,n)\,|\lambda_n|^2\right)^{\frac{\alpha s}{2}}. \qquad (46)$$

The term $\left|F_M^{(K)}(x)-F^{(K)}(x)\right|$ is more difficult:

$$\left|F_M^{(K)}(x)-F^{(K)}(x)\right|\le$$

$$\le\frac{x}{\pi}\int_0^K\left|\cos\left(\frac{xt}{2}-\frac{1}{2}\sum_{n=1}^{M}N(s,n)\arctan\frac{2t|\lambda_n|^2}{\omega_s}\right)-\right.$$

$$\left.-\frac{1}{\prod_{n=M+1}^{\infty}\left(1+\frac{4t^2|\lambda_n|^4}{\omega_s^2}\right)^{\frac{N(s,n)}{4}}}\cos\left(\frac{xt}{2}-\frac{1}{2}\sum_{n=1}^{\infty}N(s,n)\arctan\frac{2t|\lambda_n|^2}{\omega_s}\right)\right|\times$$

$$\times\frac{dt}{\prod_{n=1}^{M}\left(1+\frac{4t^2|\lambda_n|^4}{\omega_s^2}\right)^{\frac{N(s,n)}{4}}}.$$

For abbreviation we set

$$a(t):=\frac{xt}{2}-\frac{1}{2}\sum_{n=1}^{M}N(s,n)\arctan\frac{2t|\lambda_n|^2}{\omega_s}$$

$$b(t):=\frac{1}{\prod_{n=M+1}^{\infty}\left(1+\frac{4t^2|\lambda_n|^4}{\omega_s^2}\right)^{\frac{N(s,n)}{4}}}$$

$$c(t):=\frac{1}{2}\sum_{n=M+1}^{\infty}N(s,n)\arctan\frac{2t|\lambda_n|^2}{\omega_s}.$$

The condition (42) allows the estimation of $c(t)$: We have $c(t)\le c(K)$ for all $0\le t\le K$ and

$$\frac{1}{2}\sum_{n=M+1}^{\infty}N(s,n)\arctan\frac{2K|\lambda_n|^2}{\omega_s}\le\frac{1}{\omega_s}K\sum_{n=M+1}^{\infty}N(s,n)\,|\lambda_n|^2=$$

$$\frac{1}{\omega_s}\left(\sum_{n=M+1}^{\infty}N(s,n)\,|\lambda_n|^2\right)^{1-\alpha}<\frac{\pi}{2}$$

due to the assumption on M. So we have the estimation

$$\left|F_M^{(K)}(x)-F^{(K)}(x)\right|\le\frac{x}{\pi}\int_0^K\left|\cos a(t)-b(t)\cos\left(a(t)-c(t)\right)\right|dt.$$

Due to the inequality

$$|\cos a(t) - b(t)\cos(a(t) - c(t))| \leq |1 - b(t)| + |1 - \cos c(t)| + |b(t)|\,|\sin c(t)| =$$

$$= |1 - b(t)| +$$

$$+2\sin^2 \frac{c(t)^2}{2} +$$

$$+ |\sin c(t)|$$

because $0 < b(t) < 1$ and the simple estimation

$$2\sin^2 \frac{z}{2} + \sin z \leq \frac{3}{2} z$$

for $0 \leq z \leq \frac{\pi}{2}$ we have the estimation

$$\left| F_M^{(K)}(x) - F^{(K)}(x) \right| \leq \frac{x}{\pi} \int_0^K \left| 1 - \frac{1}{\prod_{n=M+1}^{\infty} \left(1 + \frac{4t^2|\lambda_n|^4}{\omega_s^2}\right)^{\frac{N(s,n)}{4}}} \right| dt +$$

$$+ \frac{3x}{4\pi} \int_0^K \sum_{n=M+1}^{\infty} N(s,n) \arctan \frac{2t|\lambda_n|^2}{\omega_s} dt \leq$$

$$\frac{x}{\pi} \int_0^K \left| 1 - \frac{1}{\prod_{n=M+1}^{\infty} \left(1 + \frac{4t^2|\lambda_n|^4}{\omega_s^2}\right)^{\frac{N(s,n)}{4}}} \right| dt + \quad (47)$$

$$+ \frac{3x}{2\pi} \left(\sum_{n=M+1}^{\infty} N(s,n) \frac{|\lambda_n|^2}{\omega_s} \right) \int_0^K t dt. \quad (48)$$

We consider the integrand of the term (47): Due to

$$|a - b| = \left| \frac{a^4 - b^4}{(a+b)(a^2 + b^2)} \right| \leq \frac{1}{4} \left| a^4 - b^4 \right|$$

for $a, b \geq 1$ we get the estimation

$$\left| 1 - \frac{1}{\prod_{n=M+1}^{\infty} \left(1 + \frac{4t^2|\lambda_n|^4}{\omega_s^2}\right)^{\frac{N(s,n)}{4}}} \right| \leq$$

$$\leq \frac{1}{4} \left| \prod_{n=M+1}^{\infty} \left(1 + \frac{4t^2|\lambda_n|^4}{\omega_s^2}\right)^{N(s,n)} - 1 \right| \leq$$

$$\leq \frac{1}{4} \sum_{m=1}^{\infty} \left(\frac{4t^2}{\omega_s^2} \right)^m \left(\sum_{n=M+1}^{\infty} |\lambda_n|^4 \, N(s,n) \right)^m .$$

This series expansion is convergent for

$$\frac{4t^2}{\omega_s^2} \sum_{n=M+1}^{\infty} |\lambda_n|^4 \, N(s,n) < 1. \tag{49}$$

for all $0 \leq t \leq K$:

$$\frac{4t^2}{\omega_s^2} \sum_{n=M+1}^{\infty} |\lambda_n|^4 \, N(s,n) \leq$$

$$\frac{4K^2}{\omega_s^2} \sum_{n=M+1}^{\infty} |\lambda_n|^4 \, N(s,n) \leq$$

$$\frac{4}{\omega_s^2} \left(\sum_{n=M+1}^{\infty} |\lambda_n|^2 \right) \left(\sum_{n=M+1}^{\infty} |\lambda_n|^2 \, N(s,n) \right)^{1-2\alpha} \leq \frac{1}{2} \tag{50}$$

due to the assumption on M. This allows the integration of the geometric series term by term and we get the estimation

$$\frac{x}{\pi} \int_0^K \left| 1 - \frac{1}{\prod_{n=M+1}^{\infty} \left(1 + \frac{4t^2 |\lambda_n|^4}{\omega_s^2} \right)^{\frac{N(s,n)}{4}}} \right| dt \leq$$

$$\frac{x}{4\pi} \sum_{m=1}^{\infty} \left(\frac{4}{\omega_s^2} \right)^m \frac{K^{2m+1}}{2m+1} \left(\sum_{n=M+1}^{\infty} |\lambda_n|^4 \, N(s,n) \right)^m \leq$$

$$\frac{xK}{4\pi} \sum_{m=1}^{\infty} \left(\frac{4}{\omega_s^2} \right)^m K^{2m} \left(\sum_{n=M+1}^{\infty} |\lambda_n|^4 \, N(s,n) \right)^m \leq$$

$$\frac{xK^3}{\omega_s^2 \pi} \left(\sum_{n=M+1}^{\infty} N(s,n) \, |\lambda_n|^2 \right) \frac{1}{1 - \frac{4K^2}{\omega_{s-1}^2} \sum_{n=M+1}^{\infty} |\lambda_n|^4 \, N(s,n)} \leq$$

$$\frac{2x}{\omega_s^2 \pi} \left(\sum_{n=M+1}^{\infty} N(s,n) \, |\lambda_n|^2 \right)^{1-3\alpha} \tag{51}$$

because of the estimation (50). The term (48) is computed:

$$\frac{3x}{2\pi} \left(\sum_{n=M+1}^{\infty} N(s,n) \frac{|\lambda_n|^2}{\omega_s} \right) \int_0^K t dt =$$

$$\frac{3x}{4\omega_s \pi} K^2 \left(\sum_{n=M+1}^{\infty} N(s,n) |\lambda_n|^2 \right) =$$

$$\frac{3x}{4\omega_s\pi}K^2\left(\sum_{n=M+1}^{\infty}N(s,n)|\lambda_n|^2\right)^{1-2\alpha}. \tag{52}$$

Collecting the results (46), (51) and (52) gives (44). □

Remark 3. *We excluded the case $s=2$. In this case we deal with an infinite weighted sum of exponential distributed random variables. A famous example is the case $\lambda_n = \frac{\pi}{n^2}$. This distribution function is closely related to the limit distribution of the classical Diaphony (see [8]). We recall this case briefly: Let $\{X_n\}_{n\geq 1}$ be independent identically distributed* $\exp(1)$ *random variables. We are interested in the distribution of*

$$Y := 2\sum_{n=1}^{\infty}\frac{1}{n^2}X_n$$

The characteristic function $\varphi(t)$ of $\frac{Y}{2}$ is given by

$$\varphi(t) = \frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{it}{n^2}\right)} = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\left(1-\frac{it}{n^2}\right)}.$$

This is the Fourier transform of

$$p(x) = 2\sum_{n=1}^{\infty}(-1)^{n-1}n^2\exp\left(-n^2x\right). \tag{53}$$

for $x>0$. Let

$$L(x) := 1-2\sum_{n=1}^{\infty}(-1)^{n-1}\exp\left(-2n^2x^2\right)$$

be the well known Kolmogorov Smirnov distribution (see [12]). Then we have

$$p(x) = \frac{d}{dx}L\left(\sqrt{\frac{x}{2}}\right)$$

and therefor

$$P(Y\leq z) = P\left(\frac{Y}{2}\leq\frac{z}{2}\right) = \int_0^{\frac{z}{2}}p(x)dx = L\left(\sqrt{\frac{z}{4}}\right)$$

Remark 4. *In the same way an approximation for an infinite weigthed sum of squares of independent $N(0,\sigma_k^2)$ random variables can be proven. Suppose that*

$$\sum_{k=1}^{\infty}|\lambda_n|^2\sigma_k^2 < \infty.$$

Then there is an index $M_0(\alpha)$ with

$$\left(\sum_{n=M+1}^{\infty}\sigma_n^2|\lambda_n|^2\right)^{1-\alpha} < \frac{1}{2\sqrt{2}}$$

for all $M > M_0(\alpha)$. The result is formulated in

Theorem 3. *Let $0 < \alpha < \frac{1}{2}$ be given and $X_k \sim N(0, \sigma_k^2)$ be independent for $k \in \mathbb{N}$ and $\{\lambda_k\}_{k\geq 1} \in \mathbb{C}$ and $M > M_0(\alpha)$. Let*

$$Y := \sum_{k=1}^{\infty} |\lambda_n|^2 X_k^2$$

and

$$Y_M := \sum_{k=1}^{M} |\lambda_n|^2 X_k^2.$$

Then we have

$$|P(Y \leq x) - P(Y_M \leq x)| \leq \frac{4}{\pi \prod_{j=1}^{3} |\sigma_n \lambda_n|} \left(\sum_{n=M+1}^{\infty} \sigma_n^2 |\lambda_n|^2 \right)^{\frac{3\alpha}{2}} +$$

$$\frac{3x}{2\alpha} \left(\sum_{n=M+1}^{\infty} \sigma_n^2 |\lambda_n|^2 \right)^{1-2\alpha} +$$

$$\frac{2x}{\pi} \left(\sum_{n=M+1}^{\infty} \sigma_n^2 |\lambda_n|^2 \right)^{2-3\alpha}$$

for $x > 0$.

References

[1] J. S. Brauchart and J. Dick. Quasi-monte carlo rules for numerical integration over the unite sphere s^2. *Numerische Mathematik*, 121:473–502, 2012.

[2] Christine Choirat and Raffaello Seri. The asymptotic distribution of quadratic discrepancies. In Harald Niederreiter and Denis Talay, editors, *Monte Carlo and Quasi-Monte Carlo Methods 2004*, pages 61–76, Berlin, Heidelberg, 2006. Springer Berlin Heidelberg.

[3] A. DasGupta. *Asymptotic Theory of Statistics and Probability.* Springer-Verlag, New York, 2008.

[4] I. T. Dimov, S. S. Stoilova, and N. Mitev. Diaphony of uniform samples over hemisphere and sphere. In Svetozar Margenov, Lubin G. Vulkov, and Jerzy Waśniewski, editors, *Numerical Analysis and Its Applications*, pages 257–264, Berlin, Heidelberg, 2009. Springer Berlin Heidelberg.

[5] R. F. Boisvert F. W. Olver, D. M. Lozier and C. W. Clark. *NIST Handbook of Mathematical Functions.* Cambridge Univerity Press, New York, 2010.

[6] A.H. Feiveson and F.C. Delaney. The distribution and properties of a weighted sum of chi squares. *National Aeronautics and Space Administration*, 1968.

[7] N. N. Lebedev. *Special functions and their applications [in Russian]*. GIFML, Moscow-Leningrad, 1963.

[8] H. Leeb. The asymptotic distribution of diaphony in one dimension. *École dees hautes études commerciales [Groupe d'études et de recherche en analyse des décisions]*, 1996.

[9] C. Müller. *Spherical harmonics*. Lecture notes in mathematics. Springer-Verlag, 1966.

[10] B. Klinger P. Grabner and R. F. Tichy. Discrepancies of point sequences on the sphere and numerical integration. *Multivariate Approximation:Recent Trends and Results*, pages 95–112, 1997.

[11] Earl D. Rainville. Notes on legendre polynomials. *Bull. Amer. Math. Soc.*, 51(4):268–271, 04 1945.

[12] A. Renyi. *Probability theory*. Akademiai Kiado, Budapest, 1970.

[13] P. Zinterhof and C. Amstler. Uniform distribution, Discrepancy and reproducing kernel Hilbert spaces. *Journal of Complexity*, 17:497–515, 2001.

[14] Peter Zinterhof. über einige Abschätzungen bei de Approximation von Funktionen mit Gleichverteilungsmethoden. *sterreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse. Sitzungsberichte. Abteilung II.*, 185, 01 1976.