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A note on a weak limit of the diaphony generated by Hermite functions

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1 Introduction and Result

Let $\lambda = (\lambda_1, \ldots, \lambda_s) \in (0,1)^s$ and $\nu = (n_1, \ldots, n_s) \in \mathbb{N}_0^s$. We use the notation $\lambda^\nu := (\lambda_1^{\nu_1}, \ldots, \lambda_s^{\nu_s})$. For $\nu \in \mathbb{N}_0$ and $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$ let

$$h_\nu(x) = h_\nu(x_1, \ldots, x_s) = \exp\left(-\frac{\|x\|_2^2}{2}\right) \prod_{j=1}^{s} \frac{H_{n_j}(x_j)}{\sqrt{2^{n_j} n_j! \sqrt{\pi}}}$$

denote the Hermite function with index ν. By $\|x\|_2$ we denote the usual Euclidian norm on $\mathbb{R}^s$. The Hermite functions form an orthonormal basis of $L^2(\mathbb{R}^s)$. We consider the following subset of $L^2(\mathbb{R}^s)$ given by

$$H_\lambda = \left\{ f(x) : \mathbb{R}^s \to \mathbb{C} | f(x) = \sum_{\nu \in \mathbb{N}_0^s} \lambda^\nu a_\nu h_\nu(x); \sum_{\nu \in \mathbb{N}_0^s} |a_\nu|^2 < \infty \right\}.$$

Equipped with the scalar product defined by

$$\langle \lambda^\nu h_\nu, \lambda^\mu h_\mu \rangle = \delta_{\mu\nu} \tag{1}$$

the space H_λ forms a reproducing kernel Hilbert space (RKHS) with kernel

$$K_\lambda(x,y) : \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}, K_\lambda(x,y) =$$

$$\prod_{j=1}^{s} \frac{1}{\sqrt{\pi}\sqrt{1-\lambda_j^4}} \exp\left(\frac{4x_j y_j \lambda_j^2 - (1+\lambda_j^4)(x_j^2 + y_j^2)}{2(1-\lambda_j^4)} \right)$$

Let $\{x_k\}_{k \geq 1} \in \mathbb{R}^s$ be a point sequence. The QMC integration error $E_N(f)$ for $f \in H_\lambda$ with respect to the normed Gaussian measure with density

$$p(x) : \mathbb{R}^s \to \mathbb{R}; p(x) = \frac{1}{\sqrt{2\pi}^s} \exp\left(-\frac{\|x\|_2^2}{2}\right)$$

is given by

$$E_N(f) = \left| \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \int_{\mathbb{R}^s} f(x)p(x)dx \right|.$$

It can be estimated by

$$E_N(f) \leq D_N(x_k). \|f\|$$

where $\|.\|$ is the norm induced by the scalar product (1). The expression $D_N(x_k)$ is independent from the integrand and depends only on the point sequence. It is the Diaphony of the point sequence $\{x_k\}_{k\geq 1}$. For the definition of the classical Diaphony see [4]. A simple computation gives an explicit formula of the square of $D_N(x_k)$:

$$(D_N(x_k))^2 = \frac{1}{N^2} \sum_{k,l=1}^{N} K_\lambda^{\sim}(x_k, x_l) \tag{2}$$

where

$$K_\lambda^{\sim}(x, y) = K_\lambda(x, y) - \frac{\exp\left(-\frac{\|x\|_2^2}{2}\right) + \exp\left(-\frac{\|y\|_2^2}{2}\right)}{\sqrt{2\pi}^s} + \frac{1}{(2\sqrt{\pi})^s}$$

and therefor

$$(D_N(x_k))^2 = \frac{1}{N^2} \sum_{k,l=1}^{N} K_\lambda(x_k, x_l) - \frac{2}{N\sqrt{2\pi}^s} \sum_{k=1}^{N} \exp\left(-\frac{\|x_k\|_2^2}{2}\right) + \frac{1}{(2\sqrt{\pi})^s} =$$

$$= \sum_{\nu\in\mathbb{N}_0^s} \lambda^{2\nu} \left(\frac{1}{N} \sum_{k=1}^{N} h_\nu(x_k) - \frac{\delta_{\nu 0}}{\sqrt{2\sqrt{\pi}}^s} \right)^2.$$

In the classical case H. Leeb [3] showed weak convergence of the classical Diaphony to a normal distribution. Let us recall briefly the classical Diaphony (see [4]): Let

$$h(x) = 1 - \frac{\pi^2}{6} + \frac{\pi^2}{2} (1 - 2\{x\})^2 = 2\pi^2 B_2(\{x\}) + 1$$

where $B_2(x)$ denotes the second Bernoulli polynomial and $\{x\}$ denotes the fractional part of x. Let

$$\{y_k\}_{k\geq 1} = \{(y_k^{(1)}, \ldots, y_k^{(s)})\}_{k\geq 1} \in [0, 1)^s$$

and

$$H : [0, 1)^s \to \mathbb{R}; H(y) = H\left(y^{(1)}, \ldots, y^{(s)}\right) = \prod_{j=1}^{s} h\left(y^j\right) - 1.$$

The classical Diaphony is given by

$$(F_N(x_k))^2 = \frac{1}{N^2} \sum_{k,l=1}^{N} H(y_k - y_l) = -1 + \frac{1}{N^2} \sum_{k,l=1}^{N} \prod_{j=1}^{s} \left(1 + 2\pi^2 B_2\left(\{y_k^{(j)} - y_l^{(j)}\}\right)\right).$$

This means that the sum over $H(x_k, x_k)$ is a constant depending on N and the dimension s. In the case of the diaphony generated by Mehlers kernel the sum over the diagonal elements is a random variable. In fact we have

$$N\left(D_N(x_k)\right)^2 = \frac{1}{N}\sum_{k,l=1}^{N} K_\lambda^\sim(x_k, x_l) = \frac{1}{N}\sum_{k=1}^{N} K_\lambda^\sim(x_k, x_k) + \frac{2}{N}\sum_{k=2}^{N}\sum_{l=1}^{k-1} K_\lambda^\sim(x_k, x_l). \tag{3}$$

The proof of Theorem 1 below is an application of the methods developed in [3], only the first summand in 3 requires special treatment. In the following I_s denotes the s-dimensional unity matrix.

Theorem 1. *Let $\lambda_j \in (-1, 1)$ with*

$$\sum_{j=1}^{\infty} |\lambda_j|^2 < \infty$$

and $x, y \sim N(0, I_s)$ independent, identically distributed random variables. Let

$$\mu_s := E\left(K_\lambda^\sim(x, x)\right) = \frac{1}{\sqrt{\pi}^s}\left(\prod_{j=1}^{s}\left(\frac{1}{\sqrt{\left(1-\lambda_j^2\right)\left(3-\lambda_j^2\right)}}\right) - \frac{1}{2^s}\right)$$

$$d_s^2 := E\left(\left(K_\lambda^\sim(x, x)\right)^2\right) - \mu_s^2 =$$

$$\frac{1}{\sqrt{\pi}^s}\prod_{j=1}^{s}\frac{1}{\sqrt{(1-\lambda_j^2)(5-3\lambda_j^2)}} + \frac{2}{(2\pi)^s}\prod_{j=1}^{s}\frac{1}{\sqrt{(1-\lambda_j^2)(3-\lambda_j^2)}} -$$

$$\frac{4}{(\sqrt{2}\pi)^s}\prod_{j=1}^{s}\frac{1}{\sqrt{(1-\lambda_j^2)}} + \frac{4}{(2\pi\sqrt{3})^s} - \frac{3}{(2\sqrt{\pi})^s} - \mu_s^2.$$

$$\sigma_s^2 := E\left(\left(K_\lambda^\sim(x, y)\right)^2\right) = \frac{1}{\pi^s}\left(\prod_{j=1}^{s}\frac{1}{\sqrt{(1-\lambda_j^4)(9-\lambda_j^4)}} - \frac{1}{2^s}\left(\frac{2}{\sqrt{3}^s} - \frac{1}{2^s}\right)\right)$$

Let $\{x_k\}_{k\geq 1} \sim N(0, I_s)$ be a sequence of independet, identically distributed random variables. We have

$$\frac{ND_N^2(x_k) - \mu_s}{\sigma_s\sqrt{2\frac{N-1}{N}}} \to_d N(0, 1)$$

provided that $N, s \to \infty$ and

$$\frac{1}{N}\max\left(\frac{d_s^2}{\sigma_s^2}, \frac{E\left(\left(K_\lambda^\sim(x, y)\right)^4\right)}{\sigma_s^4}\right) \to 0 \tag{4}$$

and

$$\prod_{j=1}^{s} \frac{\sqrt{(1-\lambda_j^4)\left(1-\frac{\lambda_j^4}{9}\right)}}{\left(1-\frac{\lambda_j^4}{3}\right)\sqrt{1+\frac{\lambda_j^4}{3}}} \to 0. \tag{5}$$

2 Technical Lemmas

We start with the following lemma which is simply proven by computing the expectation:

Lemma 1. *Let $x, y \sim N(0, I_s)$ i.i.d. random variables. Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Then we have*

$$E\left(\exp\left(-\frac{a^2}{2}\|x\|_2^2\right)\exp\left(-\frac{b^2}{2}\|y\|_2^2\right)K_\lambda^n(x,y)\right) =$$

$$\prod_{j=1}^{s} \frac{1}{\sqrt{\pi}^n \sqrt{1-\lambda_j^4}^{\,n-1}} \frac{1}{\sqrt{(n^2+(a^2+1)(b^2+1))(1-\lambda_j^4)+n(a^2+b^2+2)(1+\lambda_j^4)}}.$$

Application of the previous Lemma gives the moments of $K_\lambda^\sim(x,y)$ for $x, y \sim N(0, I_s)$:

$$E\left((K_\lambda^\sim(x,y))^n\right) =$$

$$\left(-\frac{2}{\sqrt{2\pi}^s}\right)^n n! \sum_{l=0}^{n}\left(-\frac{1}{2\sqrt{2}^s}\right)^l \frac{1}{l!} +$$

$$+\frac{n!}{(2\sqrt{2\pi})^{sn}} \sum_{i_1=1}^{n} \frac{\sqrt{4\pi}^{si_1}}{\sqrt{\pi}^{i_1 s} i_1! \prod_{j=1}^{s}\sqrt{1-\lambda_j^4}^{\,i_1-1}} \sum_{i_2=0}^{n-i_1} \frac{(-1)^{i_2}\sqrt{2}^{si_2}}{(n-i_1-i_2)!} \times$$

$$\times \sum_{l=0}^{i_2} \frac{1}{l!(i_2-l)!} \frac{1}{\prod_{j=1}^{s}\sqrt{(i_1^2+(l^2+1)((i_2-l)^2+1))(1-\lambda_j^4)+i_1(l^2+(i_2-l)^2+2)(1+\lambda_j^4)}}$$

For the following we define the random variables:

$$X_{N,k} := \sum_{l=1}^{k-1} K_\lambda^\sim(x_k, x_l) \tag{6}$$

Lemma 2. *Let $\{x_k\}_{k\geq 1}$ be a sequence of i.i.d $\sim N(0, I_s)$ random variables. We have*

1.

$$E(X_{N,k}) = 0$$

2.

$$E\left(X_{N,k}^2\right) = (k-1)E\left(\left(K_\lambda^{\sim}(x,y)\right)^2\right) = \tag{7}$$

$$= \frac{k-1}{\pi^s}\left(\prod_{j=1}^{s}\frac{1}{\sqrt{1-\lambda_j^4}\sqrt{9-\lambda_j^4}} - \frac{1}{2^s}\left(\frac{2}{\sqrt{3^s}} - \frac{1}{2^s}\right)\right) =$$

$$= \frac{k-1}{(3\pi)^s}\left(\prod_{j=1}^{s}\frac{1}{\sqrt{1-\lambda_j^4}\sqrt{1-\frac{\lambda_j^4}{9}}} - \left(\frac{\sqrt{3}}{2}\right)^s\left(2-\left(\frac{\sqrt{3}}{2}\right)^s\right)\right)$$

3. We have the estimation

$$E\left(\left(K_\lambda^{\sim}(x,y)\right)^4\right) \leq \frac{27}{\pi^{2s}}\left[\prod_{j=1}^{s}\frac{1}{\sqrt{(1-\lambda_j^4)(5-3\lambda_j^2)(5+3\lambda_j^2)}}\right.$$

$$\left. + \frac{1}{2^{2s}}\left(\frac{2}{\sqrt{5}^s} + \frac{8}{2^s\sqrt{2}^s} + \frac{6}{3^s}\right) + \frac{1}{2^{4s}}\right].$$

4. Let $k \geq 2$. Then we have

$$E\left(X_{N,k}^4\right) = E\left(\left(\sum_{l=1}^{k-1} K_\lambda^{\sim}(x_k, x_l)\right)^4\right) =$$

$$(k-1)E\left(K_\lambda^{\sim}(x,y)\right)^4 + 3(k-1)(k-2)E\left(\left(K_\lambda^{\sim}(x,y)\right)^2\left(K_\lambda^{\sim}(x,z)\right)^2\right) \leq$$

$$(k-1)(3k-5)E\left(\left(K_\lambda^{\sim}(x,y)\right)^4\right).$$

5. Let $k \geq 3$ and $l < k$. Then we have

$$E\left(X_{N,k}^2 . X_{N,l}^2\right) =$$

$$= 2(l-1)E\left(\left(K_\lambda^{\sim}(x,y)\right)^2\left(K_\lambda^{\sim}(x,z)\right)^2\right) + (k-3)(l-1)\left(E\left(\left(K_\lambda^{\sim}(x,y)\right)^2\right)\right)^2 +$$

$$2(l-1)(l-2)E\left(\left(E\left(K_\lambda^{\sim}(x,y)K_\lambda^{\sim}(x,z)|y,z\right)\right)^2\right) \leq$$

$$\leq 2(l-1)E\left(\left(K_\lambda^{\sim}(x,y)\right)^4\right) + (k-3)(l-1)\left(E\left(\left(K_\lambda^{\sim}(x,y)\right)^2\right)\right)^2 +$$

$$2(l-1)(l-2)E\left(\left(E\left(K_\lambda^{\sim}(x,y)K_\lambda^{\sim}(x,z)|y,z\right)\right)^2\right).$$

Proof. The formulas are verified by straight forward computation. The estimation

$$E\left(\left(K_\lambda^\sim(x,y)\right)^2\left(K_\lambda^\sim(x,z)\right)^2\right)\leq E\left(\left(K_\lambda^\sim(x,y)\right)^4\right)$$

is follows from Cauchy-Schwarz inequality:

$$E\left(\left(K_\lambda^\sim(x,y)\right)^2\left(K_\lambda^\sim(x,z)\right)^2\right)=$$

$$\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\exp\left(-\frac{\|y\|_2^2}{2}\right)dy\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\exp\left(-\frac{\|z\|_2^2}{2}\right)dz\times$$
$$\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\left(K_\lambda^\sim(x,y)\right)^2\left(K_\lambda^\sim(x,z)\right)^2\exp\left(-\frac{\|x\|_2^2}{2}\right)dx\leq$$
$$\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\exp\left(-\frac{\|y\|_2^2}{2}\right)dy\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\exp\left(-\frac{\|z\|_2^2}{2}\right)dz\times$$
$$\left(\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\left(K_\lambda^\sim(x,y)\right)^4\exp\left(-\frac{\|x\|_2^2}{2}\right)dx\right)^{\frac{1}{2}}\times$$
$$\left(\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\left(K_\lambda^\sim(x,z)\right)^4\exp\left(-\frac{\|x\|_2^2}{2}\right)dx\right)^{\frac{1}{2}}=$$
$$\left(\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\exp\left(-\frac{\|y\|_2^2}{2}\right)dy\left(\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\left(K_\lambda^\sim(x,y)\right)^2\exp\left(-\frac{\|x\|_2^2}{2}\right)dx\right)^{\frac{1}{2}}\right)^2\leq$$
$$\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\exp\left(-\frac{\|y\|_2^2}{2}\right)dy\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\left(K_\lambda^\sim(x,y)\right)^4\exp\left(-\frac{\|x\|_2^2}{2}\right)dx=$$
$$E\left(\left(K_\lambda^\sim(x,y)\right)^4\right).$$

□

Remark 1. *The expectation* $E\left(\left(K_\lambda^\sim(x,y)\right)^4\right)$ *is bounded for all* s *because of the square summability of the* λ_j *and by a fundamental convergence theorem of infinite products (see e.g. [1]):*

Theorem 2. *An infinite product of the form*

$$\prod_{j=1}^{\infty}(1+a_j)$$

is absolutely onvergent if and only if

$$\sum_{j=1}^{\infty}|a_j|<\infty.$$

A long computation gives the conditional expectation:

$$E\left(K_{\lambda}^{\sim}(x,y)K_{\lambda}^{\sim}(x,z)|y,z\right) =$$

$$= \prod_{j=1}^{s} \frac{1}{\pi\sqrt{(1-\lambda_j^4)(3+\lambda_j^4)}} \exp\left(\frac{8y_j z_j \lambda_j^4 - (3+\lambda_j^8)(y_j^2+z_j^2)}{2(3+\lambda_j^4)(1-\lambda_j^4)}\right) -$$

$$-\frac{1}{\left(\sqrt{2}\pi\right)^s}\left[\prod_{j=1}^{s} \frac{1}{\sqrt{3-\lambda_j^4}} \exp\left(-\frac{1+\lambda_j^4}{2(1-\lambda_j^4)} z_j^2\right) + \right.$$

$$\left.\prod_{j=1}^{s} \frac{1}{\sqrt{3-\lambda_j^4}} \exp\left(-\frac{1+\lambda_j^4}{2(1-\lambda_j^4)} y_j^2\right)\right] -$$

$$-\frac{1}{(2\pi)^s}\exp\left(-\frac{\|y\|_2^2+\|z\|_2^2}{2}\right) + \frac{1}{(2\pi)^s\sqrt{2^s}}\exp\left(-\frac{\|y\|_2^2}{2}\right) + \frac{1}{(2\pi)^s\sqrt{2^s}}\exp\left(-\frac{\|z\|_2^2}{2}\right) -$$

$$-\frac{1}{(2\pi)^s}\left(\frac{1}{2^s}-\frac{1}{3^s}\right). \quad (8)$$

The square of the conditional expectation (8) is estimated by Cauchy Schwarz inequality:

$$\left(E\left(K_{\lambda}^{\sim}(x,y)K_{\lambda}^{\sim}(x,z)|y,z\right)\right)^2 \le$$

$$\le 7E\left(\prod_{j=1}^{s} \frac{1}{\pi^2(1-\lambda_j^4)(3+\lambda_j^4)} \exp\left(\frac{8y_j z_j \lambda_j^4 - (3+\lambda_j^8)(y_j^2+z_j^2)}{(3+\lambda_j^4)(1-\lambda_j^4)}\right) + \right.$$

$$+\frac{1}{(2\pi^2)^s}\left[\prod_{j=1}^{s} \frac{1}{(3-\lambda_j^4)} \exp\left(-\frac{1+\lambda_j^4}{(1-\lambda_j^4)} z_j^2\right) + \prod_{j=1}^{s} \frac{1}{(3-\lambda_j^4)} \exp\left(-\frac{1+\lambda_j^4}{(1-\lambda_j^4)} y_j^2\right)\right] +$$

$$+\frac{1}{(2\pi)^{2s}}\exp\left(-\left(\|y\|_2^2+\|z\|_2^2\right)\right) + \frac{1}{(2\pi)^{2s}2^s}\left[\exp\left(-\|y\|_2^2\right) + \exp\left(-\|z\|_2^2\right)\right] +$$

$$\left.\frac{1}{(2\pi)^{2s}}\left(\frac{1}{2^s}-\frac{1}{3^s}\right)^2\right) =$$

$$\frac{7}{(3\pi)^{2s}}\left[\prod_{j=1}^{s} \frac{1}{\sqrt{(1-\lambda_j^4)\left(1-\frac{\lambda_j^4}{9}\right)\left(1+\frac{\lambda_j^4}{3}\right)}}\right.$$

$$+2\left(\frac{\sqrt{3}}{2}\right)^s \prod_{j=1}^{s} \frac{\sqrt{1-\lambda_j^4}}{\left(1-\frac{\lambda_j^4}{3}\right)\sqrt{1+\frac{\lambda_j^4}{3}}} +$$

$$+\left(\frac{\sqrt{3}}{2}\right)^{2s}+2\left(\frac{3^{\frac{3}{4}}}{2\sqrt{2}}\right)^{2s}+$$

$$+\left(\frac{3}{4}\right)^{2s}\left(1-\left(\frac{2}{3}\right)^{s}\right)^{2}\Bigg]. \tag{9}$$

From (7) and (9) we get the

Corollary 1. *Assuming that the sequence of $\lambda_j, j=1,2\dots$ is square summable and*

$$\prod_{j=1}^{s}\frac{\sqrt{\left(1-\lambda_j^4\right)\left(1-\frac{\lambda_j^4}{9}\right)}}{\left(1-\frac{\lambda_j^4}{3}\right)\sqrt{1+\frac{\lambda_j^4}{3}}}\to 0$$

for $s\to\infty$ we have

$$\frac{\left(E\left(K_\lambda^\sim(x,y)K_\lambda^\sim(x,z)|y,z\right)\right)^2}{\left(E\left(\left(K_\lambda^\sim(x,y)\right)^2\right)\right)^2}=\frac{\left(E\left(K_\lambda^\sim(x,y)K_\lambda^\sim(x,z)|y,z\right)\right)^2}{\sigma_s^4}\to 0.$$

for $s\to\infty$.

Proof. The estimation (9) and Application of Theorem 2 gives

$$\left(E\left(K_\lambda^\sim(x,y)K_\lambda^\sim(x,z)|y,z\right)\right)^2\le$$

$$\frac{7}{(3\pi)^s}\prod_{j=1}^{s}\frac{1}{\sqrt{\left(1-\lambda_j^4\right)\left(1-\frac{\lambda_j^4}{9}\right)}\left(1-\frac{\lambda_j^4}{3}\right)\sqrt{1+\frac{\lambda_j^4}{3}}}\times$$

$$\times\left[\prod_{j=1}^{s}\frac{1-\frac{\lambda_j^4}{3}}{\sqrt{1+\frac{\lambda_j^4}{3}}}+2\left(\frac{\sqrt{3}}{2}\right)^{s}\prod_{j=1}^{s}\left(1-\lambda_j^4\right)\sqrt{1-\frac{\lambda_j^4}{3}}+c_1\right]$$

with an absolute constant c_1. For the denominator we have

$$\left(E\left(\left(K_\lambda^\sim(x,y)\right)^2\right)\right)^2\ge c_2\frac{1}{(3\pi)^{2s}}\prod_{j=1}^{s}\frac{1}{\left(1-\lambda_j^4\right)\left(1-\frac{\lambda_j^4}{9}\right)}$$

with an absolute constant c_2 and therefor

$$\frac{\left(E\left(K_\lambda^\sim(x,y)K_\lambda^\sim(x,z)|y,z\right)\right)^2}{\left(E\left(\left(K_\lambda^\sim(x,y)\right)^2\right)\right)^2}\le$$

$$C\prod_{j=1}^{s}\frac{\frac{1}{\sqrt{\left(1-\lambda_j^4\right)\left(1-\frac{\lambda_j^4}{9}\right)}\left(1-\frac{\lambda_j^4}{3}\right)\sqrt{1+\frac{\lambda_j^4}{3}}}}{\frac{1}{\left(1-\lambda_j^4\right)\left(1-\frac{\lambda_j^4}{9}\right)}}=$$

$$= C \prod_{j=1}^{s} \frac{\sqrt{(1-\lambda_j^4)\left(1-\frac{\lambda_j^4}{9}\right)}}{\left(1-\frac{\lambda_j^4}{3}\right)\sqrt{1+\frac{\lambda_j^4}{3}}}$$

with an absolute constant C. □

Remark 2. *Condition (5) is fulfilled e.g. if* $\lambda_j = \lambda$ *for* $0 < |\lambda| < 1$.

3 Proof of Theorem 1

We have

$$ND_N^2(x_k) = \frac{1}{N}\sum_{k,l=1}^{N} K_\lambda^{\sim}(x_k, x_l) = \frac{1}{N}\sum_{k=1}^{N} K_\lambda^{\sim}(x_k, x_k) + \frac{2}{N}\sum_{k=2}^{N}\sum_{l=1}^{k-1} K_\lambda^{\sim}(x_k, x_l) =$$

$$= \frac{1}{N}\sum_{k=1}^{N} K_\lambda^{\sim}(x_k, x_k) + \frac{2}{N}\sum_{k=2}^{N} X_{N,k}$$

where $X_{N,k}$ denote the random variables defined in (6). By subtracting μ_s on both sides and multiplication by

$$\frac{1}{\sigma_s}\sqrt{\frac{N}{2(N-1)}}$$

we have

$$\frac{1}{\sigma_s}\sqrt{\frac{N}{2(N-1)}}\left(ND_n^2(x_k) - \mu_s\right) = \frac{1}{\sigma_s}\sqrt{\frac{N}{2(N-1)}}\left(\frac{1}{N}\sum_{k=1}^{N} K_\lambda^{\sim}(x_k, x_k) - \mu_s\right) +$$

$$\frac{1}{\sigma_s}\sqrt{\frac{N}{2(N-1)}}\,\frac{2}{N}\sum_{k=2}^{N} X_{N,k}.$$

Weak convergence of the left side can be shown by Slutzkys theorem (see [2]) if the first summand converges in probability to a constant and the second one converges weakly. Let $\varepsilon > 0$ be given. Application of Markov inequality gives

$$P\left(\left|\frac{1}{\sigma_s}\sqrt{\frac{N}{2(N-1)}}\left(\frac{1}{N}\sum_{k=1}^{N} K_\lambda^{\sim}(x_k, x_k) - \mu_s\right)\right| > \varepsilon\right) \leq$$

$$\frac{1}{\varepsilon^2}\frac{1}{\sigma_s^2}\frac{N}{2(N-1)}\frac{1}{N}\left(E\left(K_\lambda^{\sim}(x,x)\right)^2 - \mu_s^2\right) =$$

$$= \frac{1}{\varepsilon^2}\frac{d_s^2}{\sigma_s^2}\frac{1}{2(N-1)} \to 0$$

because

$$E\left(\left(\frac{1}{N}\sum_{k=1}^{N}K_{\lambda}^{\sim}(x_k,x_k)-\mu_s\right)^2\right)=E\left(\left(\frac{1}{N}\sum_{k=1}^{N}K_{\lambda}^{\sim}(x_k,x_k)\right)^2\right)-$$

$$-2\mu_s E\left(\frac{1}{N}\sum_{k=1}^{N}K_{\lambda}^{\sim}(x_k,x_k)\right)+\mu_s^2=$$

$$\frac{1}{N}E\left((K_{\lambda}^{\sim}(x,x))^2\right)+\frac{1}{N^2}\sum_{k,l=1;k\neq l}K_{\lambda}^{\sim}(x_k,x_k)K_{\lambda}^{\sim}(x_l,x_l)-\mu_s^2=$$

$$\frac{1}{N}E\left((K_{\lambda}^{\sim}(x,x))^2\right)+\frac{N-1}{N}\mu_s^2-\mu_s^2=$$

$$\frac{1}{N}E\left((K_{\lambda}^{\sim}(x,x))^2-\mu_s^2\right).$$

due of the independence of x_k and assumption (4). So by Slutzkys theorem the weak limit of

$$\frac{1}{\sigma_s}\sqrt{\frac{N}{2(N-1)}}\left(ND_n^2(x_k)-\mu_s\right)$$

and

$$\frac{1}{\sigma_s}\sqrt{\frac{2}{N(N-1)}}\sum_{k=1}^{N}X_{N,k}$$

coincide. From here we proceed as in [3]. Let us define the random variables

$$Z_{N,k}:=\frac{1}{\sigma_s}\sqrt{\frac{2}{N(N-1)}}X_{N,k}$$

and

$$S_{N,m}:=\sum_{k=2}^{m}Z_{N,k}.$$

and let $\sigma(x_1,x_2,\ldots,x_k)$ be the σ-algebra generated by $x_1,\ldots,x_k$. Then

$$\{S_{N,m};\sigma(x_1,x_2,\ldots,x_m)\}$$

is a zero mean, square integrable triangular martingale array. Martingale CLT ensures that

$$S_{N,N}\to_d N(0,1)$$

if we have

1. $$E\left(\max_{2\leq k\leq N}Z_{N,k}^2\right)<\infty$$

   for all N.

2.

$$\max_{2\le k\le N} |Z_{N,k}| \to_p 0$$

3.

$$\sum_{k=2}^{N} Z_{N,k}^2 \to_{L^2} 1$$

Item 1 is immedeately verified by

$$E\left(\max_{2\le k\le N} Z_{N,k}^2\right) \le \sum_{k=2}^{N} E\left(Z_{N,k}^2\right) = 1$$

for all N. Item 2 is verified by

$$P\left(\max_{2\le k\le N} |Z_{N,k}| > \varepsilon\right) \le \frac{1}{\varepsilon^4}\sum_{k=2}^{N} E\left(Z_{N,k}^4\right).$$

We get for the remaining expectation (M is an absolute constant)

$$\sum_{k=2}^{N} E\left(Z_{N,k}^4\right) \le \frac{4}{N^2(N-1)^2\sigma_s^4}\left(N^3 + MN^2\right) E\left(\left(K_\lambda^\sim(x,y)\right)^4\right)$$

$$\le M_1 \frac{1}{N}\frac{E\left(\left(K_\lambda^\sim(x,y)\right)^4\right)}{\sigma_s^4} \to 0$$

according to assumption (4). For Item 3 we get

$$E\left(\left(\sum_{k=2}^{N} Z_{N,k}^2 - 1\right)^2\right) = E\left(\left(\sum_{k=2}^{N} Z_{N,k}^2\right)^2\right) - 1 =$$

$$= \sum_{k=2}^{N} E\left(Z_{N,k}^4\right) + 2\sum_{k=3}^{N}\sum_{l=2}^{k-1} E\left(Z_{N,k}^2 Z_{N,l}^2\right) - 1.$$

From condition (2) we know that the first summand tends to 0 for $N, s \to \infty$. For the second one we have by Lemma 2

$$2\sum_{k=3}^{N}\sum_{l=2}^{k-1} Z_{N,k}^2 Z_{N,l}^2 \le M_1 \frac{E\left(\left(K_\lambda^\sim(x,y)\right)^4\right)}{N\sigma_s^4} + 1 + M_2 \frac{\left(E\left(K_\lambda^\sim(x,y)K_\lambda^\sim(x,z)|y,z\right)\right)^2}{\left(E\left(\left(K_\lambda^\sim(x,y)\right)^2\right)\right)^2} \to 1$$

for $N, s \to \infty$ according to (4) and corollary 1 where M_1, M_2 denote absolute constants .

Remark 3. *In the same way we can prove the following*

Theorem 3. *Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ with $|\lambda_j| < \frac{1}{\sqrt{3}}$ and*

$$\sum_{j\geq 1} |\lambda_j|^2 < \infty.$$

Let $x_k \sim N\left(0, \frac{1}{\sqrt{2}} I_s\right), k = 1, 2, \ldots$ i.i.d. random variables. Let

$$L_\lambda(x,y) : \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}; L_\lambda(x,y) = \prod_{j=1}^{s} \frac{1}{\sqrt{\pi}\sqrt{1-\lambda_j^4}} \exp\left(\frac{2x_j y_j \lambda_j^2 - \lambda_j^4\left(x_j^2 + y_j^2\right)}{1-\lambda_j^4}\right)$$

and

$$L_\lambda^\sim(x,y) = L_\lambda(x,y) - \frac{1}{\sqrt{\pi}^s}.$$

Let

$$F_N^2(x_k) := \frac{1}{N^2} \sum_{k,l=1}^{N} L_\lambda^\sim(x_k, x_l) = \frac{1}{N^2} \sum_{k,l=1}^{N} L_\lambda(x_k, x_l) - \frac{1}{\sqrt{\pi}^s}.$$

Let

$$M_s := E\left(L_\lambda^\sim(x,x)\right) = \frac{1}{\sqrt{\pi}^s}\left(\prod_{j=1}^{s} \frac{1}{1-\lambda_j^2} - 1\right)$$

$$V_s^2 := E\left(\left(L_\lambda^\sim(x,y)\right)^2\right) = \frac{1}{\pi^s}\left(\prod_{j=1}^{s} \frac{1}{1-\lambda_j^4} - 1\right)$$

$$D_s^2 = E\left(\left(L_\lambda^\sim(x,x)\right)^2 - M_s^2\right).$$

Then we have

$$\frac{N F_N^2(x_k) - M_s}{V_s\sqrt{2\frac{N-1}{N}}} \to_d N(0,1)$$

for $N, s \to \infty$ provided that

$$\frac{1}{N} \max\left(\frac{D_s^2}{V_s^2}, \frac{E\left(\left(L_\lambda^\sim(x,y)^4\right)\right)}{V_s^4}\right) \to 0.$$

Remark 4. *The condition $|\lambda_j| < \frac{1}{\sqrt{3}}$ arises from the fact that $EL_\lambda(x,x)^2$ does not exist for $|\lambda_j| \geq \frac{1}{\sqrt{3}}$.*

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